CS5314 Randomized Algorithms

Homework 2
Due: November 7, 2013 (before class)

Announcement: Quiz 1 is on November 12, 2013!

1. A monkey types on a 26-letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters, what is the expected number of times the word "proof" appears?

2. Suppose that we independently roll two standard six-sided dice. Let X_1 be the number that shows on the first die, X_2 be the number that shows on the second die, and X be the sum of the numbers on the two dice.

   (a) What is $\mathrm{E}[X \mid X_1 \text{ is even}]$?

   (b) What is $\mathrm{E}[X \mid X_1 = X_2]$?

   (c) What is $\mathrm{E}[X_1 \mid X = 9]$?

   (d) What is $\mathrm{E}[X_1 - X_2 \mid X = k]$, for k in the range $[2, 12]$?

3. For a coin that comes up heads independently with probability p on each flip, what is the expected number of flips until the kth head appear?

4. We roll a standard fair die over and over. What is the expected number of rolls until the first pair of consecutive sixes appear? (*Hint:* The answer is not 36.)

5. You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for this position. When you interview the candidates, you can give each of them a score, so that the one with the highest score will be the best, and there will be no ties.

   You interview the candidates one by one. Because of your company's hiring policy, after you interview the kth candidate, you either offer the candidate the job immediately, or you will forever lose the chance to hire that candidate.

   We suppose that the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings.

   Consider the following strategy: First interview m candidates but reject them all. Then, from the $(m+1)$th candidate, hire the first candidate who is better than *all* of the previous candidates you have interviewed.[1]

   (a) Let E be the event that we hire the best candidate, and let E_i be the event that the ith candidate is the best and we hire him. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

[1] That is, you will hire the kth candidate if $k > m$ and this candidate is better than all of the $k - 1$ candidates you have already interviewed.

(b) Bound $\sum_{j=m+1}^{n} \frac{1}{j-1}$ to obtain

$$\frac{m}{n}(\log_e n - \log_e m) \leq \Pr(E) \leq \frac{m}{n}(\log_e(n-1) - \log_e(m-1)).$$

(c) Show that $m(\log_e n - \log_e m)/n$ is maximized when $m = n/e$. Explain why this means $\Pr(E) \geq 1/e$ for this choice of m.

6. (Tutorial: No Marks) Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

   (a) What is the probability that $X = Y$?

   (b) What is $\Pr(\min(X, Y) = k)$?

   (c) What is $\mathrm{E}[\max(X, Y)]$ and $\mathrm{E}[\min(X, Y)]$?