Randomized algorithm

Tutorial 6
Solution for Assignment 3

Hint for Assignment 4

Solution for assignment 3

Exercise 1

- Let *X* be a Poisson random variable with mean μ.
  a) What is the most likely value of *X* when
     λ is an integer?
     λ is not an integer?

Exercise 1

- [Sol]

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\Pr(X = k+1) = \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!}$$

$$\frac{\Pr(X = k+1)}{\Pr(X = k)} = \frac{\lambda}{k+1}$$

Exercise 1

$$k < \lambda \rightarrow \Pr(X = k+1) > \Pr(X = k)$$

$$k = \lambda \rightarrow \Pr(X = k+1) = \Pr(X = k)$$

$$k > \lambda \rightarrow \Pr(X = k+1) < \Pr(X = k)$$

The most likely value of X is when $X = \lfloor \lambda \rfloor$.

If λ is an integer, both λ and $\lambda - 1$ are the most likely values.

Exercise 1

b) We define the median of X to be the least number m such that $\Pr(X \leqq m) \geqq 1/2$. What is the median of X when $\lambda = 3.9$?

[Sol]

$$\Pr(X = 0) = 0.020$$

$$\Pr(X = 1) = 0.079 \Rightarrow 0.099$$

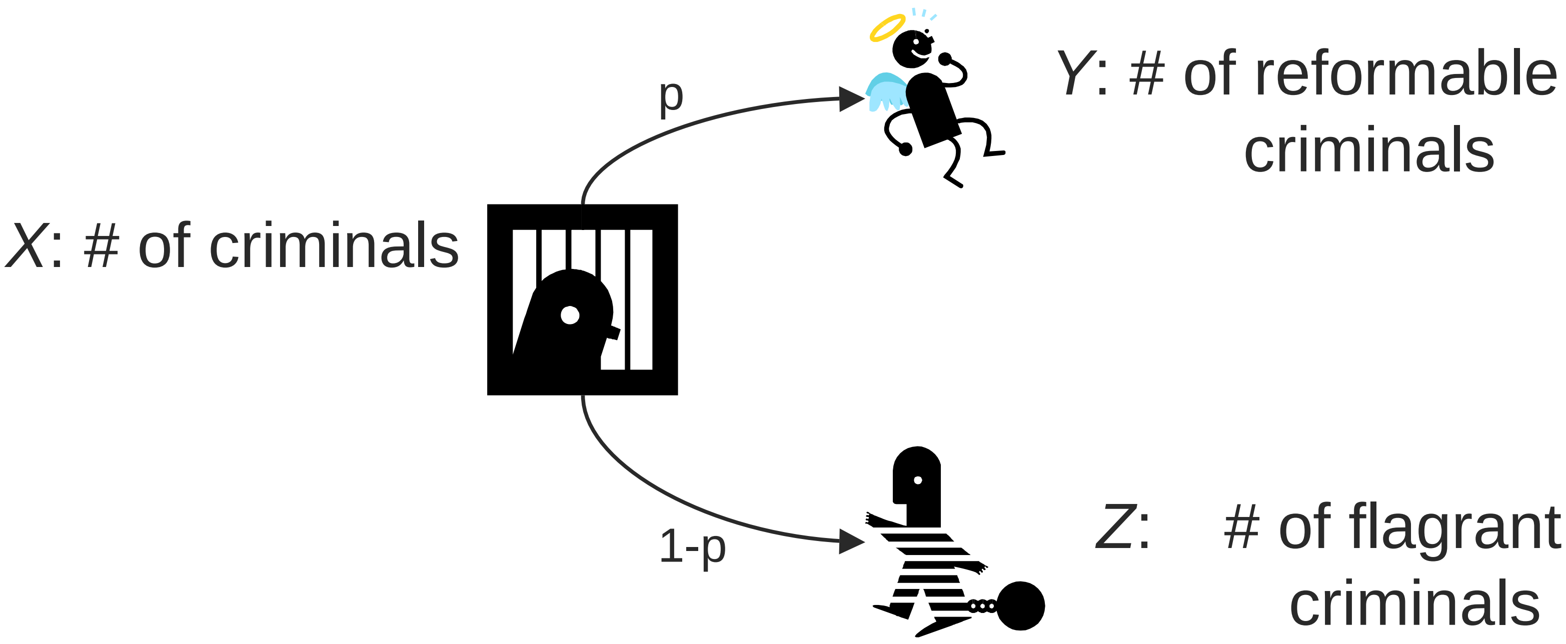
$$\Pr(X = 2) = 0.154 \Rightarrow 0.253$$

$$\Pr(X = 3) = 0.200 \Rightarrow 0.453$$

$$\Pr(X = 4) = 0.195 \Rightarrow 0.648$$

Exercise 2

- X: Poisson random variable(μ)



Exercise 2

- Are *Y* and *Z* independent?
- [Sol]

$$\Pr(Y=k)$$

$$=\sum_{m=k}^{\infty}\Pr(X=m)\Pr(Y=k\mid X=m)$$

$$=\sum_{m=k}^{\infty}\frac{\lambda^m e^{-\lambda}}{m!}\binom{m}{k}p^k(1-p)^{m-k}$$

$$=\frac{(\lambda p)^k e^{-p\lambda}}{k!}$$

Exercise 2

$$\Pr(Y = k \text{ and } Z = j)$$

$$= \Pr(X = k + j)\Pr(Y = k \mid X = k + j)$$

$$= \frac{\lambda^{k+j}e^{-\lambda}}{(k+j)!}\binom{k+j}{k}p^k(1-p)^j$$

$$= \frac{(\lambda p)^k e^{-p\lambda}}{k!}\cdot\frac{(\lambda(1-p))^j e^{-(1-p)\lambda}}{j!}$$

$$= \Pr(Y = k)\Pr(Z = j)$$

Exercise 3

a) Now, *b* balls are in play.
f(*b*): the expected number of balls that survive to the subsequent round.
Given an explicit formula for *f*(*b*).

$$\Pr(i\text{th bin has exactly 1 ball}) = b\frac{1}{n}(1-\frac{1}{n})^{b-1}$$

$$\text{E[number of bins with 1 ball]} = n\Pr(i\text{th}\ldots) = b(1-\frac{1}{n})^{b-1}$$

$$f(b) = b - \text{E[number of bins with 1 ball]} = b(1-(1-\frac{1}{n})^{b-1})$$

Exercise 3

b) Show that $f(b) \leqq b^2/n$.

$$\text{By Bernoulli's inequality}: (1-\frac{1}{n})^{b-1} \geq 1-\frac{b-1}{n}$$

$$f(b) = b(1-(1-\frac{1}{n})^{b-1})$$

$$\leq b - b(1-\frac{b-1}{n})$$

$$= \frac{b(b-1)}{n} \leq \frac{b^2}{n}$$

Exercise 3

c) Suppose that every round the number of balls served was exactly the expected value.

   Show that all the balls would be served in O(log log n) rounds.

Exercise 3 : Solution

- Suppose we have n / k balls initially, for some fixed constant $k > 1$.

  From part (b),

  $$f(n/k) \le n / k^2.$$

  After r rounds,

  $$f^{(r)}(n/k) \le n / k^s \qquad \text{where } s = 2^r$$

  When $r = \log_k \log_2 n = O(\log \log n)$,

  $$f^{(r)}(n/k) \le 1$$

Exercise 3

Now, consider about the case that we have *n* balls initially.

If *n* is large, #balls after 1st round is :

$$\lim_{n\to\infty} n(1-(1-\frac{1}{n})^{n-1})$$

$$=\lim_{n\to\infty} n(1-(1-\frac{1}{n-1})^{n-1})$$

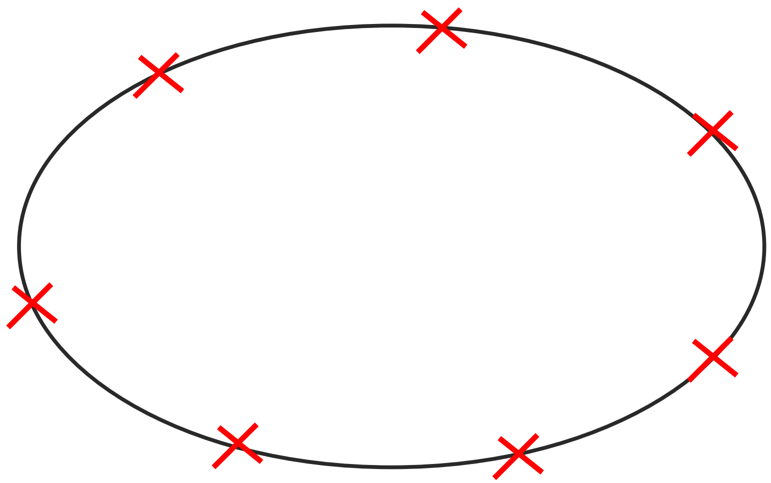
$$=n(1-\frac{1}{e})$$

Exercise 4

a) Argue that the maximum load in this case is only $O(\log \log n / \log \log \log n)$ with probability that approaches 1 as $n \to \infty$.

Exercise 4

- Among all *n* bins, we choose log *n* bins (evenly) as representatives.



Exercise 4 : Solution

$$\Pr(\exists \text{ bin} \geq 2M \text{ balls})$$

$$\leq \Pr(\exists \text{ Rep} \geq M)$$

$$\leq (\log n)(\frac{1}{M!})$$

$$\leq \frac{1}{\log n} \qquad (\text{if } M = \frac{\log\log n}{\log\log\log n})$$

$$\to 0 \qquad (\text{when } n \to \infty)$$

Exercise 5

Consider *n* balls thrown randomly into *n* bins

- Let $X=X_1+X_2+\ldots+X_n$, where

  $X_i = 1$ if *i*-th bin is empty ; 0 otherwise.

- Let $Y=Y_1+Y_2+\ldots+Y_n$, where

  each Y_i is an independent Bernoulli random variables with

$$\Pr(Y_i = 1) = (1-1/n)^n.$$

Exercise 5

a) Show that $E[X_1 X_2 \ldots X_k] \leqq E[Y_1 Y_2 \ldots Y_k]$.

$$E[X_1 X_2 \ldots X_k] = \Pr(X_1 = 1 \cap X_2 = 1 \cap \ldots \cap X_k = 1) = \left(\frac{n-k}{n}\right)^n$$

$$E[Y_1 Y_2 \ldots Y_k] = \Pr(Y_1 = 1 \cap Y_2 = 1 \cap \ldots \cap Y_k = 1) = \left(1 - \frac{1}{n}\right)^{kn}$$

$$\left(\frac{n-k}{n}\right)^n \leq \left(1 - \frac{1}{n}\right)^{kn}$$

Exercise 5(a) : Alternative solution

- By induction

$$
\begin{aligned}
&\mathrm{E}[X_1 X_2 \ldots X_k X_{k+1}] \\
&= \mathrm{E}[X_1 X_2 \ldots X_k | X_{k+1} = 1] \Pr(X_{k+1} = 1) \\
&\le \mathrm{E}[X_1 X_2 \ldots X_k] \Pr(X_{k+1} = 1) \\
&\le (1 - \frac{1}{n})^{(k+1)n} \\
&= \mathrm{E}[Y_1 Y_2 \ldots Y_k Y_{k+1}]
\end{aligned}
$$

Exercise 5

b) Show that $X_{i1}^{k1} X_{i2}^{k2} \cdots X_{ij}^{kj} = X_{i1} X_{i2} \cdots X_{ij}$

X_i is an indicator

Thus,

$$X_{i1}^{k1} X_{i2}^{k2} \ldots X_{ij}^{kj} = 1 = X_{i1} X_{i2} \ldots X_{ik}$$

or

$$X_{i1}^{k1} X_{i2}^{k2} \ldots X_{ij}^{kj} = 0 = X_{i1} X_{i2} \ldots X_{ik}$$

Exercise 5

c) Show that $E[e^{tX}] \leqq E[e^{tY}]$

$$E[e^{tX}]$$

$$= E[1 + tX + \frac{(tX)^2}{2!} + \ldots]$$

$$= E[1] + tE[X] + t^2 E[\frac{X^2}{2!}] + \ldots$$

$$E[X] \leq E[Y] \qquad \text{(by (a))}$$

Exercise 5

$$\mathrm{E}[X^2] = \mathrm{E}[(X_1 + X_2 + \ldots + X_n)^2]$$

$$= \mathrm{E}[X_1^{\ 2} + 2(X_1X_2 + X_1X_3 + \ldots + X_1X_n) + X_2^{\ 2} + \ldots + X_n^{\ 2}]$$

$$\leq \mathrm{E}[Y_1^{\ 2} + 2(Y_1Y_2 + Y_1Y_3 + \ldots + Y_1Y_n) + Y_2^{\ 2} + \ldots + Y_n^{\ 2}]$$

$$= \mathrm{E}[Y^2]$$

Exercise 5

$$\mathrm{E}[e^{tX}]$$

$$= \mathrm{E}[1] + t\mathrm{E}[X] + t^2\mathrm{E}[\frac{X^2}{2!}] + \ldots$$

$$\le \mathrm{E}[1] + t\mathrm{E}[Y] + t^2\mathrm{E}[\frac{Y^2}{2!}] + \ldots$$

$$= \mathrm{E}[e^{tY}]$$

Exercise 5

d) Derive a Chernoff bound for

$$\Pr(X \geqq (1+\delta)\mathrm{E}[X])$$

$$\mathrm{E}[e^{tY}] = \Pi \mathrm{E}[e^{tY_i}]$$

$$= [(1-p) + pe^t]^n$$

$$= [1 + p(e^t - 1)]^n$$

$$\leq \mathrm{e}^{\mathrm{np}(\mathrm{e}^{\mathrm{t}} - 1)}$$

Exercise 5

$$\Pr(X \geq (1+\delta)\mathrm{E}[X])$$

$$= \Pr(e^{tX} \geq e^{t(1+\delta)\mathrm{E}[X]})$$

$$\leq \frac{E[e^{tX}]}{e^{t(1+\delta)\mu}} \leq \frac{E[e^{tY}]}{e^{t(1+\delta)\mu}} \qquad (\text{Set } \mathrm{E}[X] = \mu)$$

$$= \frac{e^{\mu(e^t - 1)}}{e^{t(1+\delta)\mu}} \qquad (\text{Choose } t = \ln(1+\delta))$$

$$\leq \frac{e^{\mu\delta}}{(1+\delta)^{(1+\delta)\mu}}$$

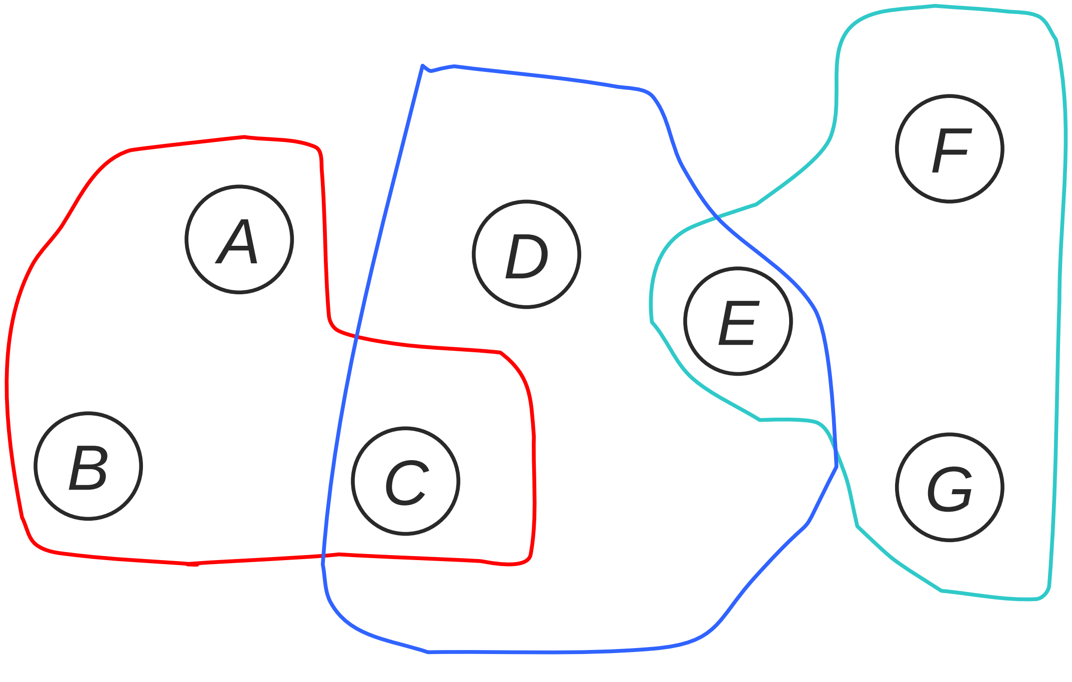
Hint for assignment 4

Exercise 1

- *k*-uniform hypergraph

  V = {A,B,C,D,E,F,G}

  E = { {ABC}, {CDE}, {EFG} }



Exercise 1

- Given a k-uniform hypergraph with

$$E = \{S_1, S_2, S_3, \ldots, S_{|C|}\} ,$$

$$|C| \leqq 4^{k-1}-1, \quad \text{and} \quad k \geqq 2.$$

Show that there exists a 4-coloring such that no k-set is monochromatic.

[Hint] You can do it without any hint.

Exercise 2

- Anti-chain
  F, a family of subsets of *N*={1,2,…,*n*} is called *anti-chain* if there are no *A*, *B* in *F* satisfying *A* in *B*.

  Ex: F={ {1,3,4} , {2,4} , {1,5} , {6} }

  What if F={ {1,3,4}, {2,4} , {1,4} , {6} }?

Exercise 2

- Let σ be a random permutation of the elements of N and consider the random variable
  $X = |\{i : \{\sigma(1), \sigma(2), \ldots, \sigma(i)\} \in F\}|$

  F={ {1,3}, {2}} $\sigma = (2,3,1)$
  X= 1   (why?)

  F={ {1,3}, {2}} $\sigma = (3,2,1)$
  X= 0   (why?)

Exercise 2

- Considering the expectation of *X*, prove that

$$|F| \le \binom{n}{\lfloor n/2 \rfloor}$$

Exercise 2

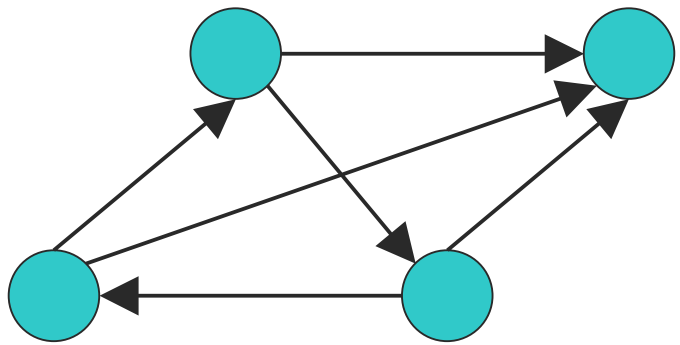
- [Hint]

  Separate F by the size of elements.
  Number of size-1 set: K_1
  Number of size-2 set K_2, …,
  Number of size-n set K_n.

  $|F| = K_1+K_2+\ldots+K_n$.
  $E[X] = ?$

Exercise 3

- Tournament
  A complete oriented graph i.e., a graph in which every pair of nodes is connected by a single uniquely directed edge.



Exercise 3

- Show that there is a tournament T with n vertices which contains at least $n!\,2^{-(n-1)}$ Hamiltonian paths.

  [Hint] You can do it without any hint.

Exercise 4

- Consider a graph in $G_{n,p}$, with $p = 1/n$. Let X be the # of triangles in the graph. Show that
  - (a) $\Pr(X \geqq 1) \leqq 1/6$
  - (b) $\lim_{n\to\infty} \Pr(X \geq 1) \geq 1/7$

[Hint] For (b), use conditional expectation inequality.

Exercise 5

- Use the general form of the Lovasz local lemma to prove that the symmetric version can be improved

  where we can replace the condition $4dp \leqq 1$ by the weaker condition $ep(d+1) \leqq 1$.

Exercise 5

- [Hint]

  1. Set x_i =1/(d+1) to the general case Lovasz local lemma.
  2. Pr(E_i) $\leqq p$ (symmetric version)
     Try to prove

$$\Pr(E_i) \leqq x_i \prod_{(i,j) \in E} (1 = x_j).$$