Randomized algorithm

Tutorial 4
Hint for Homework 3

Exercise 1

- Let X be a Poisson random variable with mean λ
  a) What is the most likely value of X when
     λ is an integer?
     λ is not an integer?

  [Hint]

  Compare $\Pr(X=k+1)$ with $\Pr(X=k)$

Exercise 1

b) We define the median of X to be the least number m such that $\Pr(X \leqq m) \geqq 1/2$. What is the median of X when $\lambda = 3.9$?

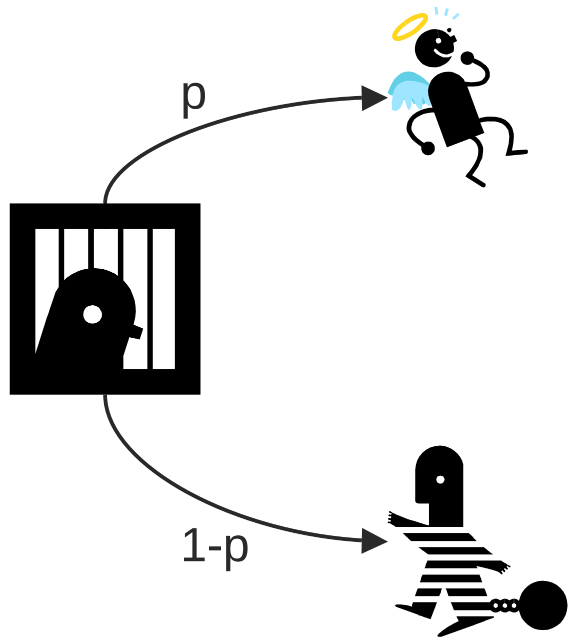
[Hint]

You may calculate it directly.

Exercise 2

- X: Poisson random variable(μ)

X: # of criminals



Y: # of reformable criminals

Z: # of flagrant criminals

Exercise 2

- Show that Y and Z are independent Poisson random variables

  [Hint]

  By definition of Poisson random variable with some condition

Exercise 2

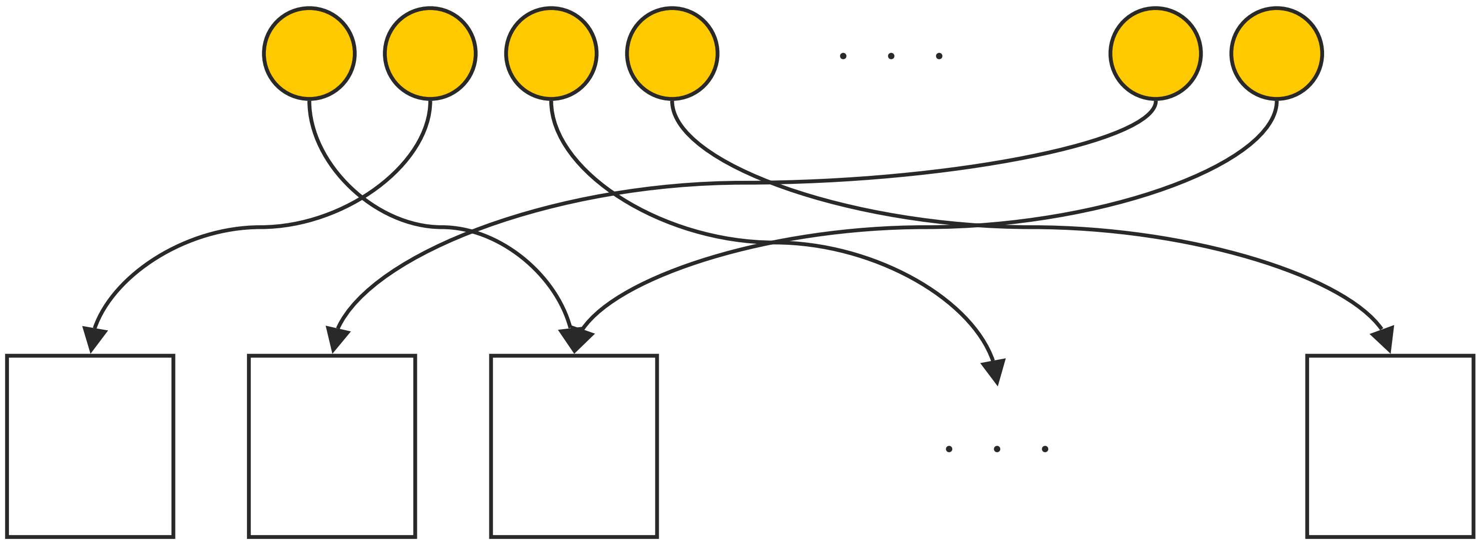
For example:

1. $\Pr(Y=k)$
   /* under what situation would Y
   be equal to k ? */

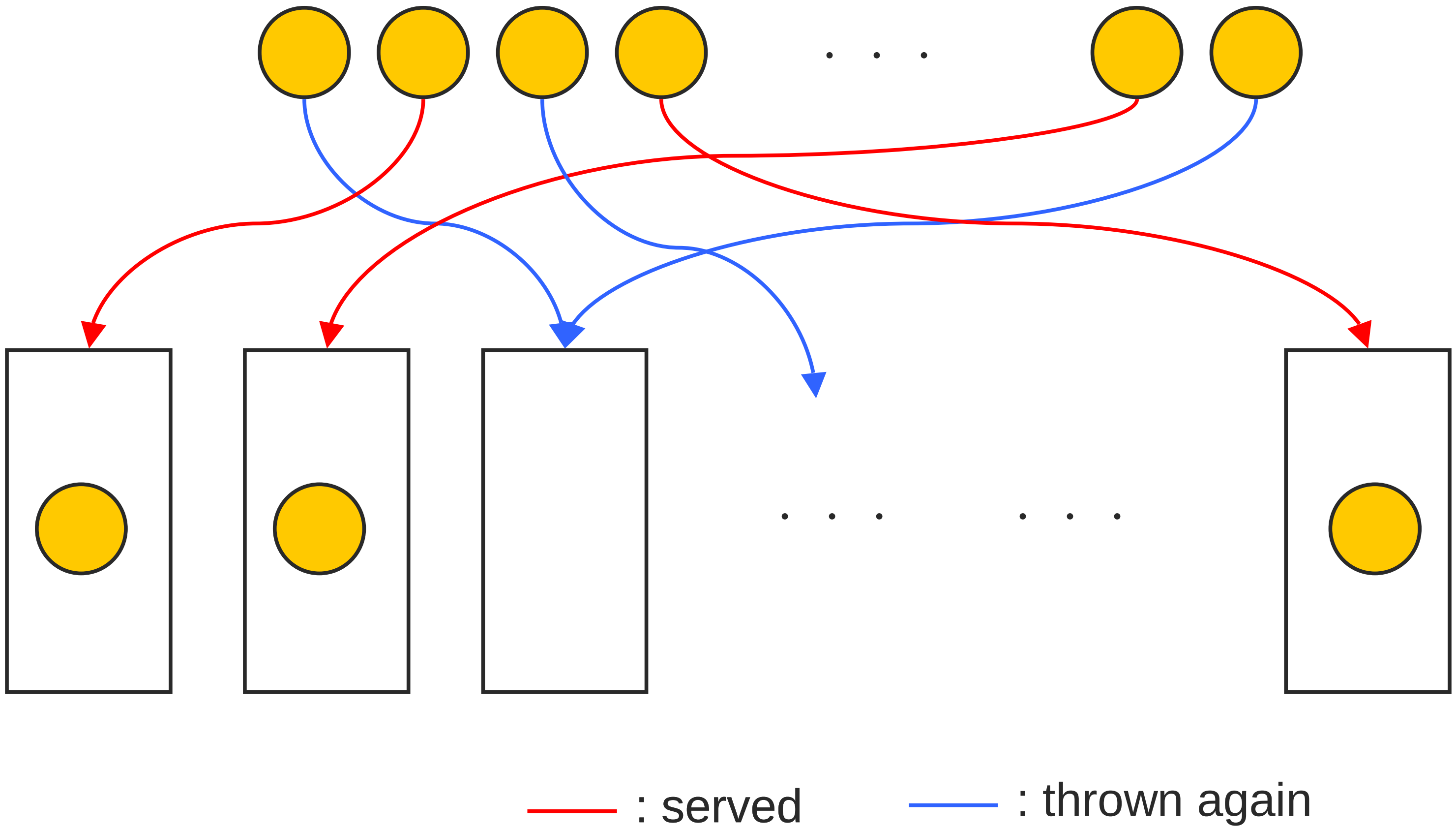
2. $\Pr(Y=k_1 \cap Z=k_2) = \Pr(Y=k_1)\Pr(Z=k_2)$?

Exercise 3

- We begin with *n* balls in the first round



Exercise 3



Exercise 3

- We finish when every ball is served.

Exercise 3

a) Suppose *b* balls are now in play.
f(*b*): the expected number of balls that survive to the subsequent round

Give an explicit formula for *f*(*b*).

[Hint]
E[# bins with 1 ball]=?

Exercise 3

b) Show that $f(b) \leq b^2/n$.

[Hint] (Bernoulli's inequality)

For all $x \geq -1$ and r in N

$$(1 + x)^r \geq 1 + rx$$

Exercise 3

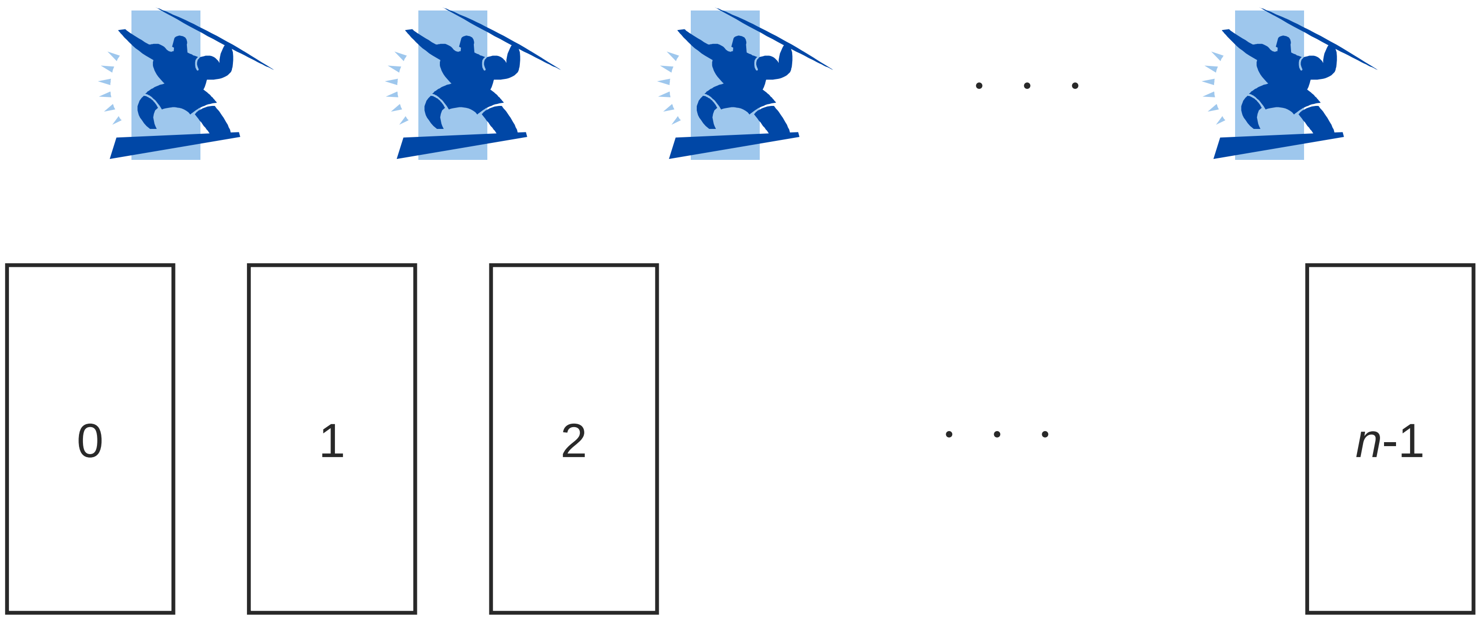
c) Suppose that every round the number of balls served was exactly the expected value. Show that all the balls would be served in O(log log n) rounds.

[Hint]
balls at each round decreased exactly according to expectation: m, $f(m)$, $f(f(m))$, $f(f(f(m)))$, …

Exercise 4

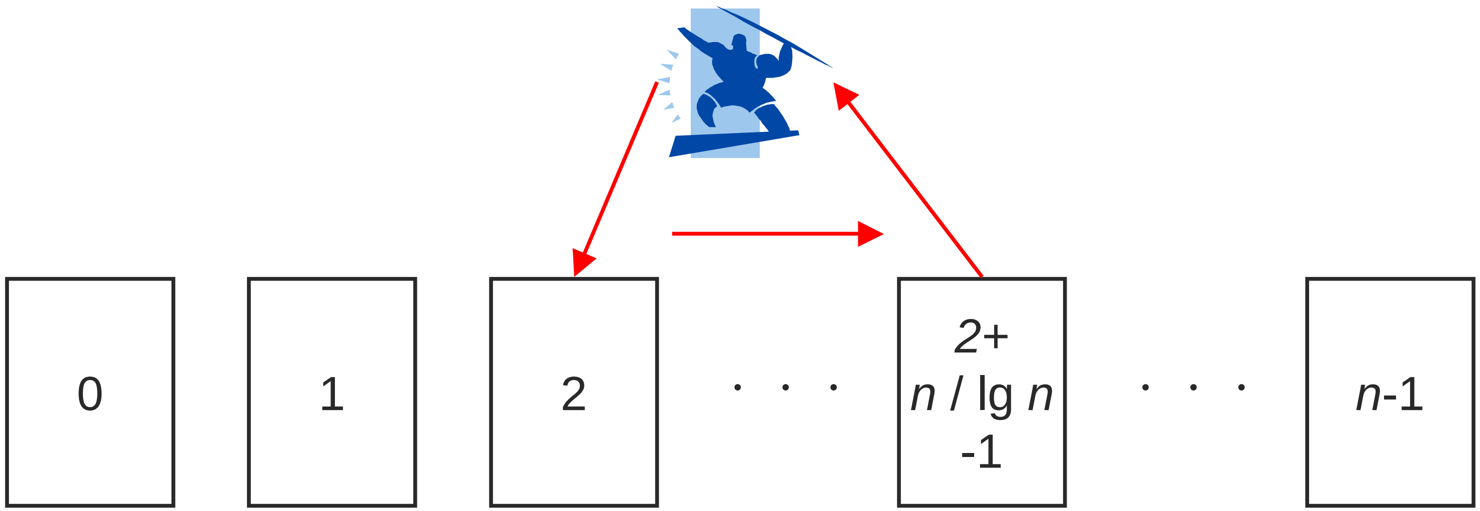
- Another kind of balls-and-bins problem

  We have $\log_2 n$ players.



Exercise 4

Each player randomly chooses a starting location ℓ and places one ball in bin ℓ mod n, ℓ +1 mod n, …, ℓ +n / $\log_2 n$ -1 mod n



Exercise 4

- Show that the maximum load is only $O(\log \log n / \log \log \log n)$ with probability approaching 1 as $n \to \infty$

[Hint]

Total number of balls = n

What is

Pr(Bin 1 receives at least M balls)?

Exercise 5

We throw *n* balls randomly to *n* bins

- Let $X = X_1 + X_2 + \ldots + X_n$
  where
  $X_i = 1$ if i^{th} bin is empty;
  $X_i = 0$ otherwise
- Let $Y = Y_1 + Y_2 + \ldots + Y_n$
  where each Y_i is an independent Bernoulli random variables with $\Pr(Y_i = 1) = (1-1/n)^n$

Exercise 5

a) Show that $E[X_1X_2\ldots X_k] \leqq E[Y_1Y_2\ldots Y_k]$.
[Hint]
By induction.

b) Show that $X_1^{k1} X_2^{k2} \cdots X_j^{kj} = X_1 X_2 \cdots X_j$

c) Show that $E[e^{tX}] \leqq E[e^{tY}]$

d) Derive a Chernoff bound for

$$\Pr(X \geqq (1+\delta)E[X])$$