Randomized algorithm

Tutorial 1

Hint for homework 1

Outline

- Lookup table problem (Exercise 1.18)
- Convex function (Exercise 2.10)
- Hint for homework 1

Lookup table problem

Lookup table problem

- F:{0,…,n-1}→{0,…,m-1}.
- For any x and y with $0 \leq x, y \leq n$-1:
- $F((x + y) \bmod n) = (F(x) + F(y)) \bmod m$.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	0	2	4	6	1	3	5	0	2	4

n=10, m=7

Lookup table problem

- Now, someone has changed exactly 1/5 of the lookup table entries

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	0	2	4	2	1	3	2	0	2	4

Lookup table problem

- (a) Given any input z, outputs the correct value $F(z)$ with probability at least 1/2. (it needs to work for every value)
  1. Pick a x, uniformly at random from $[0,\ldots,n-1]$.
  2. Compute $y = z - x \bmod n$.
  3. output $(F(x)+F(y)) \bmod m$ as our computed value for $F(z)$.

Lookup table problem

- Pr (the output $F(z)$ is correct)

  $\geqq$ Pr ($F(x)$ and $F(y)$ are not modified)

  $=$ 1-Pr ($F(x)$ or $F(y)$ are modified)

  $\geqq$ 1- (Pr ($F(x)$ is modified) +

  Pr ($F(y)$ is modified))

  $\geqq$ 1-(1/5 + 1/5) = 3/5

Lookup table problem

- (b) Can you improve the probability of returning the correct value with repeating your algorithm three times?
  1. Pick three values (with replacement) of y_1, y_2, y_3 as y values.
  2. compute the corresponding values of $F(z_1),F(z_2),F(z_3)$.
  3. If two or more of the $F(z)$ values are equal, we output such value. Otherwise, we output any one of these values.

Lookup table problem

- Pr (the output $F(z)$ is correct)

  $\geqq$ Pr (at least two of the $F(z_i),F(z_2),F(z_3)$ are correct)

  $=$ Pr (exactly two of the $F(z_1),F(z_2),F(z_3)$ are correct)+Pr (all $F(z_1),F(z_2),F(z_3)$ are correct)

  $\geqq$ 3*3/5*3/5*2/5+3/5*3/5*3/5

  $=$ 54/125+27/125 $=$ 81/125

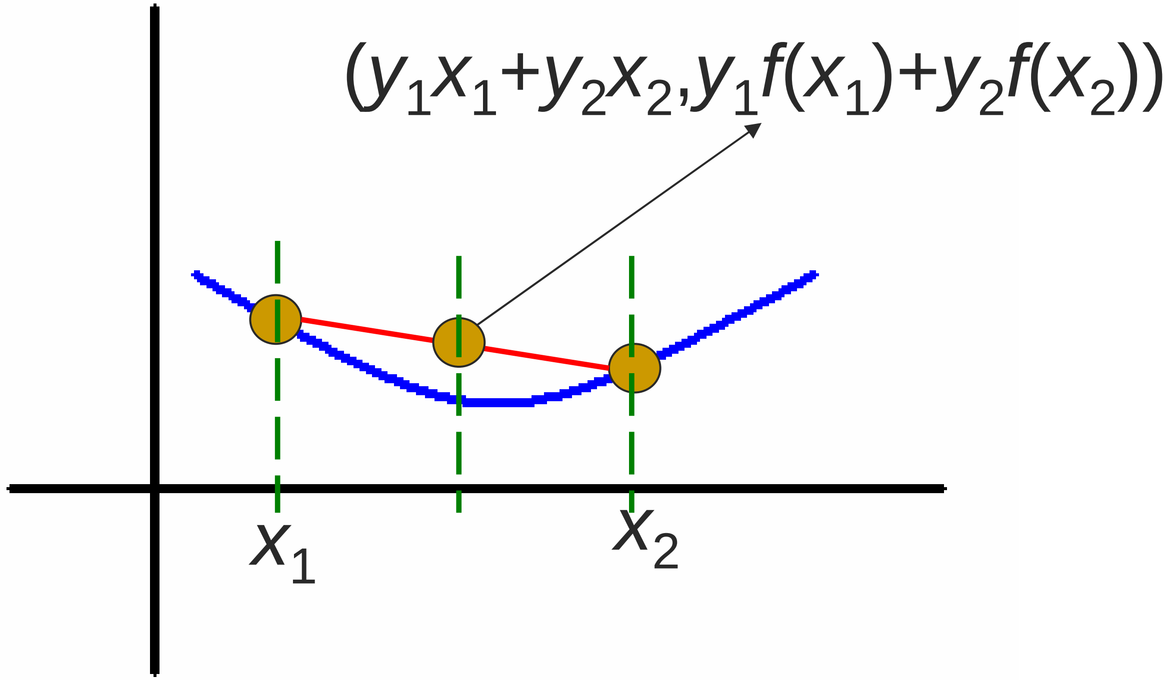
Convex function

Convex function

- Show by induction that if f:R→R is convex then, for any $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with $\sum_{i=1}^{n} y_i = 1$

$$f\left(\sum_{i=1}^{n} y_i x_i\right) \leq \sum_{i=1}^{n} y_i f(x_i)$$

Convex function



$$f(y_1x_1 + y_2x_2) \leq y_1 f(x_1) + y_2 f(x_2)$$

Convex function

$$f(y_1x_1 + y_3x_3 + y_4x_4)$$

$$= f(y_1x_1 + (y_3 + y_4)(\frac{y_3x_3}{y_3 + y_4} + \frac{y_4x_4}{y_3 + y_4}))$$

$$\leq y_1 f(x_1) + (y_3 + y_4)(\frac{y_3 f(x_3)}{y_3 + y_4} + \frac{y_4 f(x_4)}{y_3 + y_4})$$

$$= y_1 f(x_1) + y_3 f(x_3) + y_4 f(x_4)$$

Convex function

- Use former inequality to prove that if f:R→R is convex then E[*f*(*X*)] ≧ *f*(E[*X*])

$$X = x_1, x_2, \ldots, x_k$$

$$y_i = \Pr(X = x_i)$$

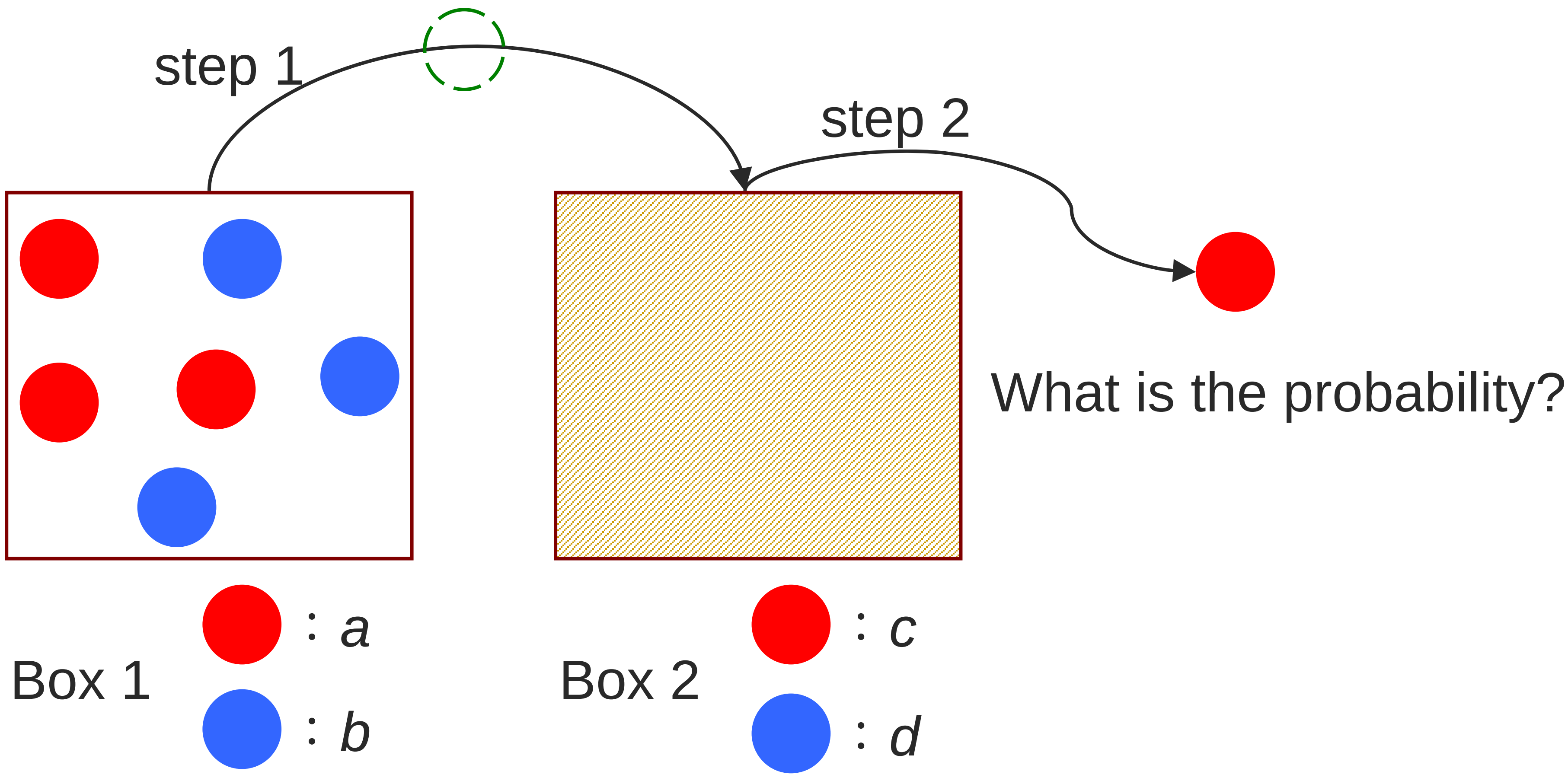
$$1.\ \mathrm{E}[X] = \sum_1^k \Pr(X = x_i) x_i = \sum_1^k y_i x_i$$

$$2.\ \mathrm{E}[f(X)] = \sum_1^k \Pr(X = x_i) f(x_i) = \sum_1^k y_i f(x_i)$$

$$f(\mathrm{E}[X]) \le \mathrm{E}[f(X)]$$

Hint for homework 1

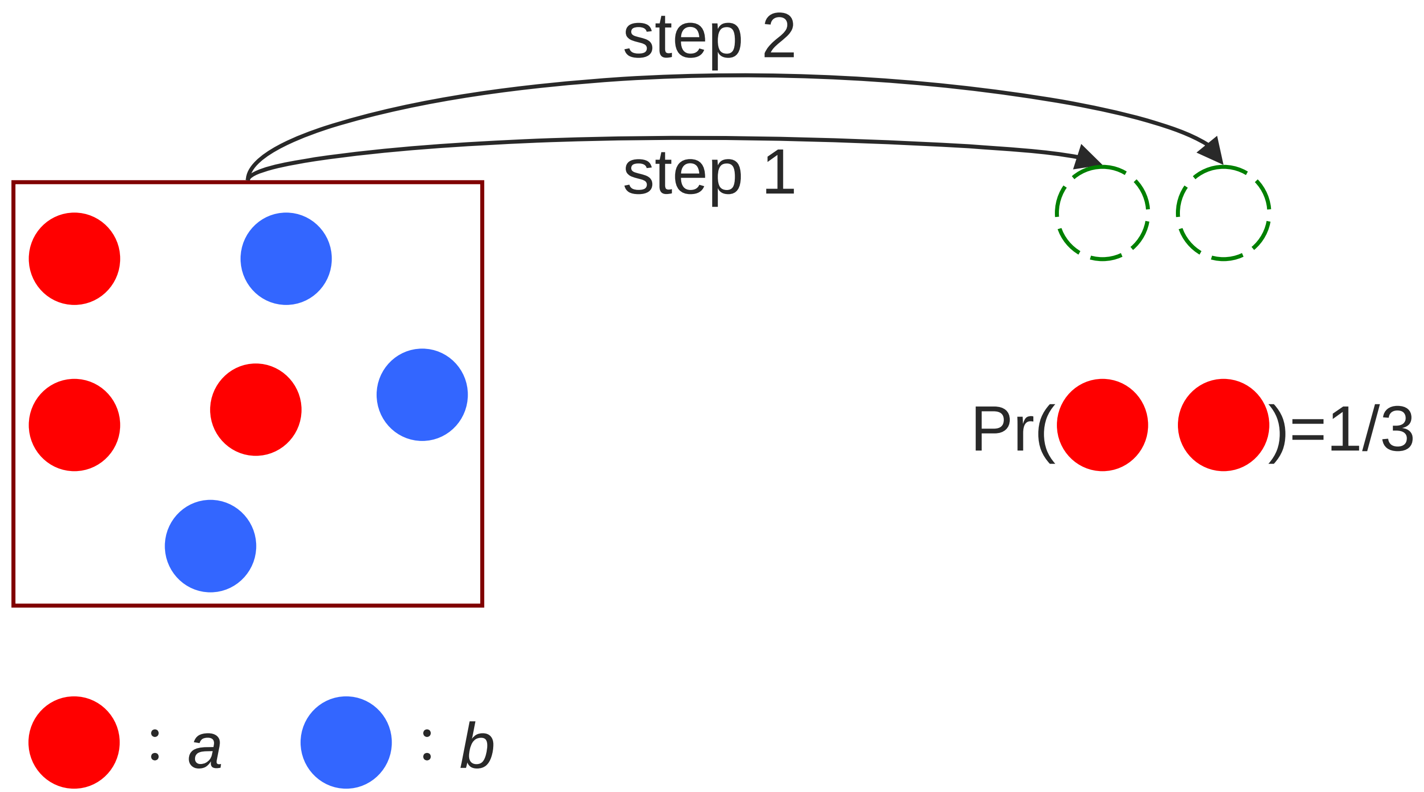
Homework 1-1



Homework 1-1

- Hint: There are two cases
  - The ball from one to the other is black.
  - The ball from one to the other is white.

Homework 1-2



The probability we get a white ball at the first time is *a*/*a*+*b*.

Homework 1-3

- Give an example of three random events *X*, *Y*, *Z* for which any pair are independent but all three are not mutually independent.

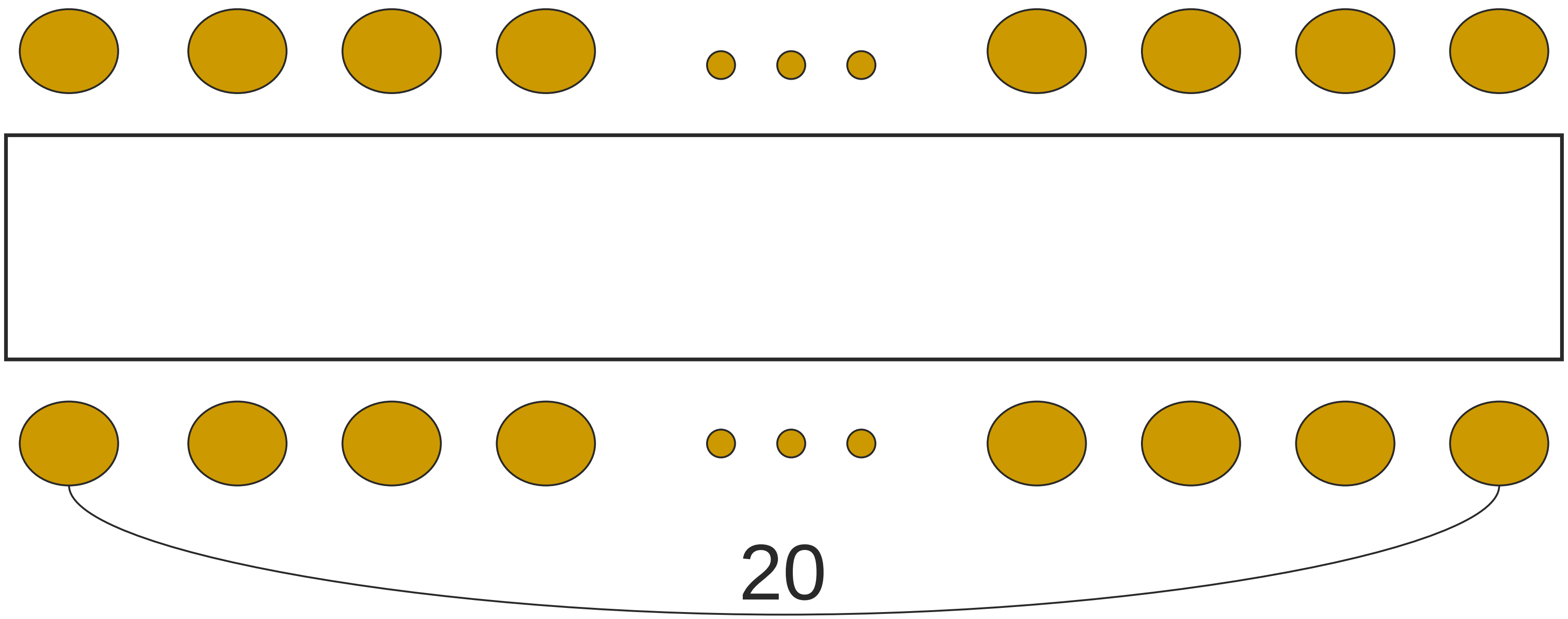
- Hint: Two dices with *X*=sum is even.

  You may use $\Pr(\bigcap_{i\in I} E_i) = \prod_{i\in I} \Pr(E_i)$ to check your answer.

Homework 1-4

- There can be at most $n(n-1)/2$ distinct min-cut sets.
- Hint: [Theorem 1.8] The algorithm outputs a min-cut set with probability at least $2/n(n-1)$.

Homework 1-5



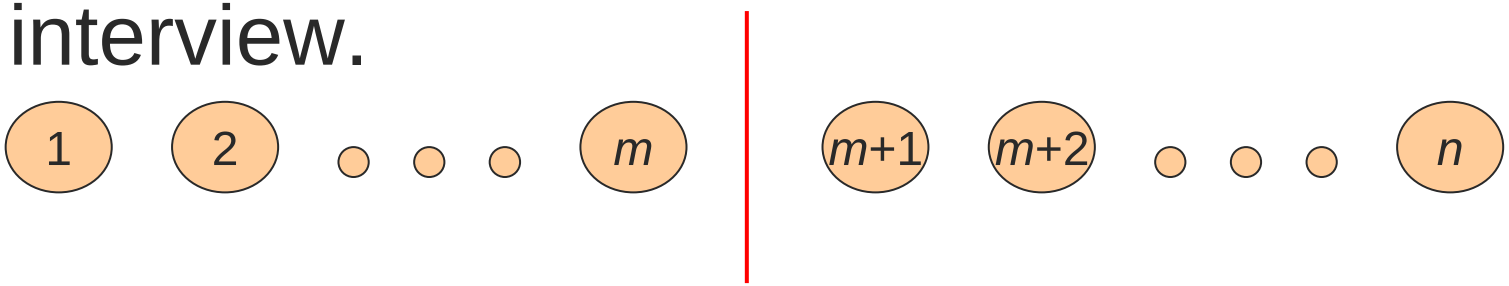
- Hint: Indicator variable/linearity of expectation.

Homework 1-6

- (a) What is the probability that $X = Y$?
- (b) What is E[max(X,Y)]?
  - min(X,Y) is the random variable denoting the number of steps you see the first head. X+Y-min(X,Y) = max(X,Y).
  - Memory-less property of geometric random variable.

Homework 1-7

- First, we give everyone a number card. The number card means the order of interview.



- Choose the best grade from 1 to m : A.
- If someone after m better than A, accept him. Otherwise, we choose the last one.

Homework 1-7

- Sometimes, we may not get the best candidate.
  - The best one's number card is less than $m+1$.
  - A, the best grade we chose from 1 to m, is not too good.
- What is a "nice" m?
  - The second best candidate is in $1 \sim m$.