# CS5314 Randomized Algorithms

Homework 1
Due: 3:20 pm, March 20, 2008 (before class)

1. (10%) There are two boxes. Suppose Box 1 contains $a$ white balls and $b$ black balls, and Box 2 contains $c$ white balls and $d$ black balls. One ball of unknown color is transferred from Box 1 into Box 2. Now, if we draw a ball from Box 2, what is the probability that it will be a white ball?

2. (20%) A box contains white and black balls. When two balls are drawn without replacement, suppose the probability that both are white is 1/3.

   (a) Show that
   $$\left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2.$$

   (b) Using part (a), show that
   $$\frac{(\sqrt{3}+1)b}{2} < a < 1 + \frac{(\sqrt{3}+1)b}{2}.$$

   (c) Find the smallest number of balls in the box.

   (d) How small can the total number of balls be if black balls are even in number?

3. (10%) Give an example of three random events $X, Y, Z$ for which any pair are independent but all three are not mutually independent.

4. (10%) There may be several different min-cut sets in a graph. Using the analysis of the randomized min-cut algorithm, argue that there can be at most $n(n-1)/2$ distinct min-cut sets.

5. (10%) Twenty couples are invited to a party. They are asked to be seated at a long table with 20 seats each side with husbands sitting at one side and wives the other side. If the seating is done at random, what is the expected number of married couples that are seated face to face?

6. (20%) Let $X$ and $Y$ be independent geometric random variables, where $X$ has parameter $p$ and $Y$ has parameter $q$.

   (a) What is the probability that $X = Y$?

   (b) What is $\mathrm{E}[\max(X, Y)]$?

7. (20%) You need a new staff assistant, and you have $n$ people to interview. You want to hire the best candidate for this position. When you interview the candidates, you can give each of them a score, with the highest score will be the best and no ties being possible.

   You interview the candidates one by one. Because of your company's hiring policy, after you interview the $k$th candidate, you either offer the candidate the job immediately, or you will forever lose the chance to hire that candidate.

We suppose that the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings.

Consider the following strategy. First, interview $m$ candidates but reject them all. Then from the $(m+1)$th candidate, hire the first one who is better than *all* of the previous candidates you have interviewed.[1]

(a) Let $E$ be the event that we hire the best assistant, and let $E_i$ be the event that $i$th candidate is the best and we hire him. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

(b) Bound $\sum_{j=m+1}^{n} \frac{1}{j-1}$ to obtain

$$\frac{m}{n}(\log_e n - \log_e m) \leq \Pr(E) \leq \frac{m}{n}(\log_e(n-1) - \log_e(m-1)).$$

(c) Show that $m(\log_e n - \log_e m)/n$ is maximized when $m = n/e$. Explain why this means $\Pr(E) \geq 1/e$ for this choice of $m$.

8. (Bonus: 10%) Let $S$ be a probability space and let $X$ and $Y$ be two different random variables on $S$. Let $Z = \min\{X, Y\}$ be random variable defined by $Z(s) = \min\{X(s), Y(s)\}$ for every $s \in S$. Disprove the following claim:

$$\mathrm{E}[Z] = \min\{\mathrm{E}[X], \mathrm{E}[Y]\}.$$

[1] That is, you will hire the $k$th candidate if $k > m$ and this candidate is better than all of the $k-1$ candidates you have already interviewed.