CS 2336
Discrete Mathematics

Lecture 10

Sets, Functions, and Relations: Part II

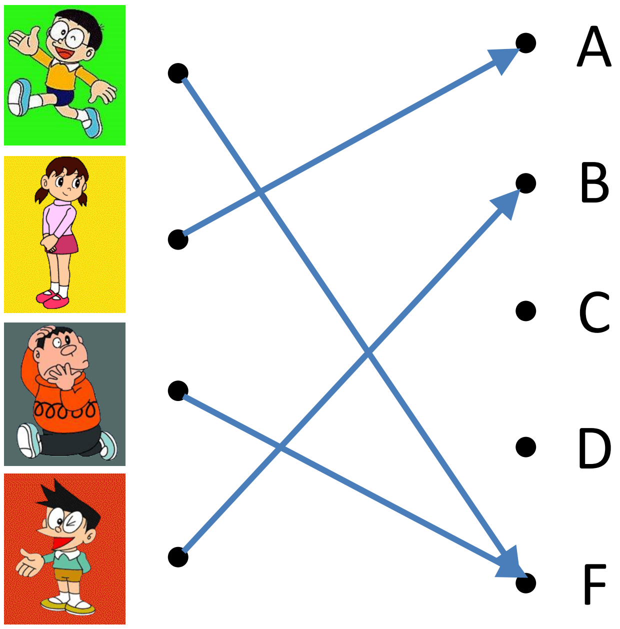
Outline

- What is a Function ?
- Types of Functions
- Floor and Ceiling Functions
- An Interesting Result

What is a Function ?

- Suppose that each student in a mathematics class will be assigned a grade A, B, C, D, F
- Let us say the grades for the students are :

Nobita (F), Shizuka (A), Takeshi (F), Suneo (B)



What is a Function ?

- The previous is an example of a function

Let A and B be two nonempty sets

A function f from A to B is an assignment of exactly one item of B to each item of A.

This relationship is denoted by

$$f : A \rightarrow B$$

We write f(a) = b if b is the unique item of B assigned by f to the item a of A, and we say b is the image of a

Terminology

- Function is also called mapping or transformation
- Given a function f : A → B

  A is called the domain

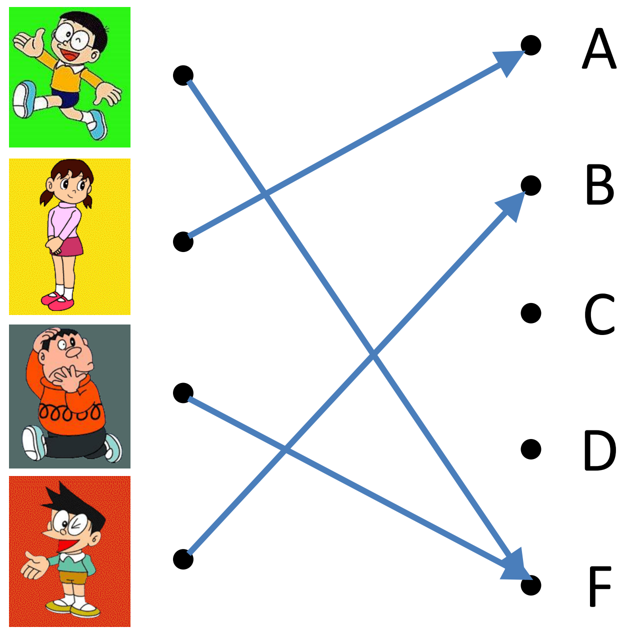
  B is called the codomain

  The subset of B that contains all images,

  $$\{ b \mid f(a) = b \text{ for some } a \text{ in } A \},$$

  is called the range

Test Your Understanding



- What is the domain, codomain, and range in the above function ?

Types of Functions

A function from A to B is said to be one-to-one, or injective, or an injection, if no two items of A have the same image in B

- Which of the following are one-to-one functions ?
  - $f : Z \rightarrow Z$, with $f(x) = x^2$
  - $g : N \rightarrow Z$, with $g(x) = x^2$

Types of Functions

A function from A to B is said to be onto, or surjective, or a surjection, if every item of B is the image of at least one item of A

Equivalently, when range equals to codomain

- Which of the following are onto functions?
  - $f : Z \rightarrow Z$, with $f(x) = 2x$
  - $g : R \rightarrow R$, with $g(x) = 2x$

Types of Functions

A function is said to be one-to-one onto, or bijective, or a bijection, if it is both one-to-one and onto

- Which of the following are bijections?
  - $f : Z \rightarrow Z$, with $f(x) = 2x$
  - $g : R \rightarrow R$, with $g(x) = 2x$

Some Counting Problems

- Let A and B be two sets, with |A| = m, |B| = n

- Problems :
  1. How many distinct injections from A to B ?
  2. How many distinct surjections from A to B ?
  3. How many distinct bijections from A to B ?

Floor and Ceiling Functions

- Let x be a real number

The floor function of x, denoted by $\lfloor x \rfloor$, is the largest integer that is smaller than or equal to x

The ceiling function of x, denoted by $\lceil x \rceil$, is the smallest integer that is larger than or equal to x

- Examples:

$\lfloor 0.5 \rfloor = 0$, $\lceil 0.5 \rceil = 1$, $\lfloor -1.1 \rfloor = -2$, $\lceil -1.1 \rceil = -1$

$\lfloor 7 \rfloor = 7$, $\lceil 7 \rceil = 7$, $\lfloor -4 \rfloor = -4$, $\lceil -4 \rceil = -4$

Floor and Ceiling Functions

- Some useful properties (n is an integer):

1. $\lfloor x \rfloor = n \Leftrightarrow n \leq x < n + 1$
2. $\lceil x \rceil = n \Leftrightarrow n - 1 < x \leq n$
3. $\lfloor x \rfloor = n \Leftrightarrow x - 1 < n \leq x$
4. $\lceil x \rceil = n \Leftrightarrow x \leq n < x + 1$
5. $\lfloor -x \rfloor = -\lceil x \rceil$
6. $\lceil -x \rceil = -\lfloor x \rfloor$

Challenges

- Which of the following are correct ?
  1. $\lfloor x + n \rfloor = \lfloor x \rfloor + n$,   where n is an integer
  2. $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$
  3. $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 0.5 \rfloor$

- Let z be a positive real. What is the maximum integer k such that

$$\lfloor kz \rfloor \leq n,$$

when n is a positive integer ?

Answers to the Challenges

- 1 is correct.

  Proof : Suppose $\lfloor x \rfloor = m$, where m = integer.

  $\Rightarrow m \le x < m + 1$

  $\Rightarrow m + n \le x + n < m + n + 1$

  $\Rightarrow \lfloor x + n \rfloor = m + n = \lfloor x \rfloor + n$

- 2 is wrong. Counter-example: $x = 0.5$, $y = 0.5$

Answers to the Challenges

- 3 is correct.

Proof : Consider the value $\{ x \} = x - \lfloor x \rfloor$. Then,

$$\text{LHS} = \lfloor 2 \lfloor x \rfloor + 2 \{ x \} \rfloor = 2 \lfloor x \rfloor + \lfloor 2 \{ x \} \rfloor$$

There are two cases :

(i) $0 \leq \{ x \} < 0.5 \Rightarrow 0 \leq 2\{ x \} < 1$

$\Rightarrow \text{LHS} = 2 \lfloor x \rfloor = \text{RHS}$

(ii) $0.5 \leq \{ x \} < 1 \Rightarrow 1 \leq 2\{ x \} < 2$

$\Rightarrow \text{LHS} = 2 \lfloor x \rfloor + 1 = \text{RHS}$

Answers to the Challenges

- To find the maximum integer k with $\lfloor kz \rfloor \leq n$, such a k must be the maximum integer with

$$kz < n + 1, \text{ or } k < (n + 1) / z$$

- There are two cases :

(i) if $(n + 1) / z$ is not an integer

$$\Rightarrow k = \lfloor (n + 1) / z \rfloor = \lceil (n + 1) / z \rceil - 1$$

(ii) if $(n + 1) / z$ is an integer

$$\Rightarrow k = (n + 1) / z - 1 = \lceil (n + 1) / z \rceil - 1$$

So it is always true that $k = \lceil (n + 1) / z \rceil - 1$

An Interesting Result

- For a positive real number z, we define the spectrum of z,

  $\mathrm{Spec}(z) = \{ \lfloor z \rfloor, \lfloor 2z \rfloor, \lfloor 3z \rfloor \lfloor 4z \rfloor, \ldots \}$

- Examples :

  $\mathrm{Spec}(\sqrt{2}) = \{ 1, 2, 4, 5, 7, 8, 9, 11, 12, 14, \ldots \}$

  $\mathrm{Spec}(2+\sqrt{2}) = \{ 3, 6, 10, 13, 17, 20, \ldots \}$

- Anything special about the above spectrums ?

An Interesting Result

- Examples :

Let $\varphi = (1 + \sqrt{5}) / 2 = 1.6180339887...$

$= \text{golden ratio}$

Then $\varphi^2 = \varphi + 1 = 2.6180339887...$

$\text{Spec}(\varphi) = \{ 1, 3, 4, 6, 8, 9, 11, 12, 14, 16, ... \}$

$\text{Spec}(\varphi^2) = \{ 2, 5, 7, 10, 13, 15, 18, ... \}$

- Anything special about the above spectrums ?

An Interesting Result

- In general, we have the following theorem :

Let α and β be two positive irrational numbers such that

$$1/\alpha + 1/\beta = 1.$$

Then, the spectrums

$$\text{Spec}(\alpha) \text{ and } \text{Spec}(\beta)$$

cover all the positive integers, and they have no common items