CS 2336
Discrete Mathematics

Lecture 8

Counting: Permutations and Combinations

Outline

- Definitions
- Permutation
- Combination
- Interesting Identities

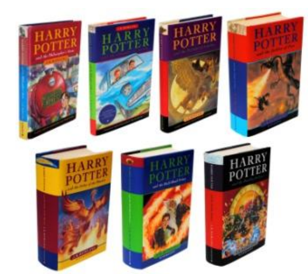
Definitions

- Selection and arrangement of objects appear in many places

  ➔ We often want to compute # of ways to select or arrange the objects

- Examples :
  1. How many ways to select 2 people from 5 candidates ?

  2. How many ways to arrange 7 books on the bookshelf ?



Definitions

- In most textbooks, we use the word

combination $\Leftrightarrow$ selection

An **r-combination of n objects** is an unordered selection of r objects from the n objects

- Example :

{ c, d } is a 2-combination of { a, b, c, d, e }

Definitions

- In most textbooks, we use the word

permutation $\Leftrightarrow$ arrangement

An **r-permutation of n objects** is an ordered arrangement of r objects from the n objects

- Example :

cabd is a 4-permutation of { a, b, c, d, e }

Definitions

- Further, we define the following notation:

C(n, r) denotes the number of r-combinations of n distinct objects

P(n, r) denotes the number of r-permutations of n distinct objects

- What are the values of C(n, n), C(n, 1), C(3, 2), and P(3, 2) ?

Test Your Understanding

- Why are the following equalities correct ?

1. $P(n, r) = P(r, r) \times C(n, r)$
2. $P(n, n) = P(n, r) \times P(n - r, n - r)$
3. $C(n, r) = C(n, n - r)$

Permutation

- In fact, there is a formula for P(n, r) :

$$P(n, r) = n\,(n-1)(n-2)\ \dots\ (n-r+1)$$

- Proof :

P(n, r) = # ways to get r of n objects in some order.

There are n ways to choose the 1st object,

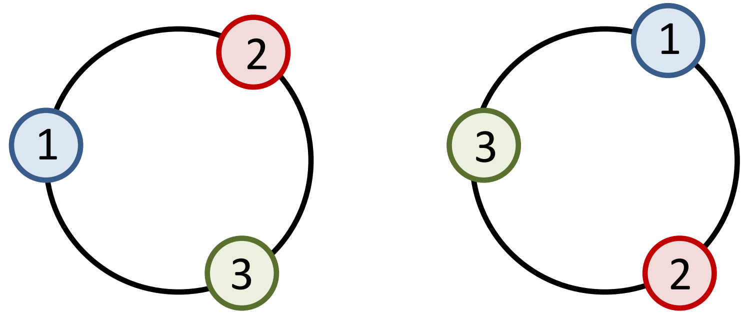
$n - 1$ ways to choose the 2nd object, ... ,

$n - r + 1$ ways to choose the rth object

➔ Result follows from rule of product

Examples

- Ex 1 : How many ways to select a first-prize, a second-prize, and a third-prize winners from 100 different people ?
- Ex 2 : How many ways can n people be ordered to form a ring ?



The above are considered the same
(as relative order is the same)

With Indistinguishable Objects

- How many different strings can be made by re-ordering the letters of the word “SUCCESS” ?

- Answer :

  First, suppose that all the 7 letters are distinct. Then, there will be 7! different strings.

  Now, if we make the two Cs indistinguishable, we will only have 7!/2! different strings.

  Further, if the three Ss are indistinguishable, the number of different strings becomes (7!/2!)/3!

With Indistinguishable Objects

- In general, if there are n objects, with

  n_1 indistinguishable objects of type 1,

  n_2 indistinguishable objects of type 2,

  … ,

  n_k indistinguishable objects of type k,

➔ the number of n-permutations is :

$$\frac{n!}{n_1!\, n_2! \dots n_k!}$$

Examples

- If we have 5 dashes and 8 dots, how many different ways to arrange them ?

· · · _ _ _ · _ · _ · · ·

- If we can only use 7 symbols of them, how many different arrangements are there ?

· _ · _ · _ ·

Examples

- Show that for any positive integer k,

$$(k!)! \text{ is divisible by } k!^{(k-1)!} ?$$

- For instance, when k = 3,

$$(k!)! = (3!)! = 6! = 720$$

$$k!^{(k-1)!} = (3!)^{2!} = 6^2 = 36$$

With Unlimited Repetitions

- Suppose that there are n distinct objects, each with unlimited supply

- How many r-permutations are there ?

  That is, how many ways to get a total of r objects from them, and then form an arrangement ?

- Answer : n^r

Examples

- Ex 1 : Consider all numbers between 1 and 10^{10}
  (i) How many of them contain the digit 1 ?
  (ii) How many of them do not ?

- Ex 2 :
  (i) How many bit strings of length n are there ?
  (ii) How many contain even number of 0s?

Combination

- Recall that

$$P(n, r) = P(r, r) \times C(n, r)$$

- Thus, we have

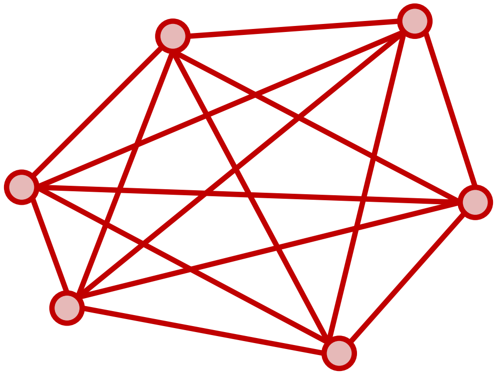
$$C(n, r) = n\,(n-1)(n-2)\ \ldots\ (n-r+1)\ /\ r!$$

- Alternatively, we can express C(n, r) as :

$$C(n, r) = n!\ /\ (\ (n-r)!\ r!\ )$$

Examples

- Consider a hexagon where no three diagonals meet a one point



- How many diagonals are there ?
- How many intersections between the diagonals ?
- How many line segments are the diagonals divided by their intersections ?

Examples

- In how many ways can we select 3 numbers from 1, 2, …, 300, such that their sum is a multiple of 3 ?
- Hint :

  When the sum is a multiple of 3, what special property does the 3 numbers have ?
- Answer : $100^3 + 3 \times C(100, 3)$

Examples

- Five pirates have discovered a treasure box

  They decided to keep the box in a locked room, so that all the locks of the room can be opened if and only if 3 or more pirates are present

- How to do so ? How many locks do they need ?

  (Each pirate may possess keys to different locks)

With Unlimited Repetitions

- Suppose that there are n distinct objects, each with unlimited supply

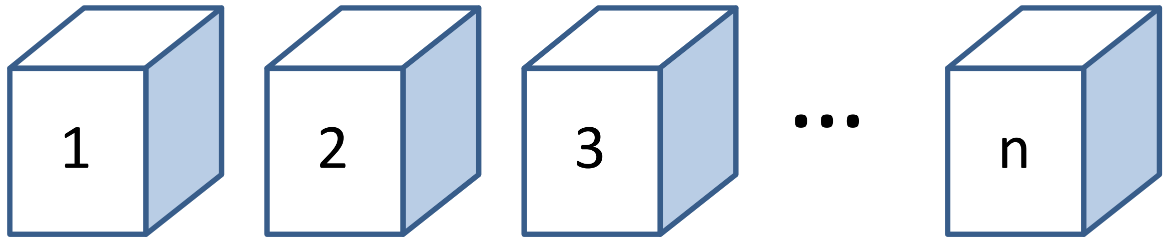
- How many r-combinations are there ?

  That is, how many ways to get a total of r objects from them, and the ordering is not important ?

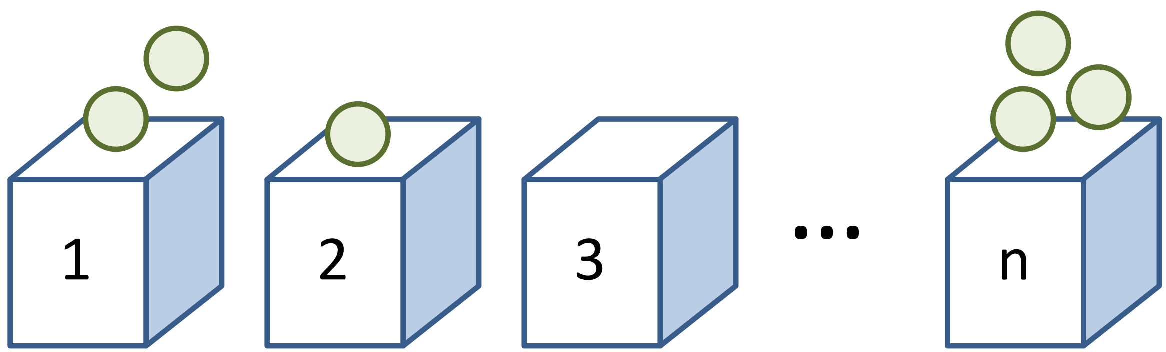
- Answer : $C(n - 1 + r, r)$ [Why?]

With Unlimited Repetitions

- Imagine we have a box for each type of objects

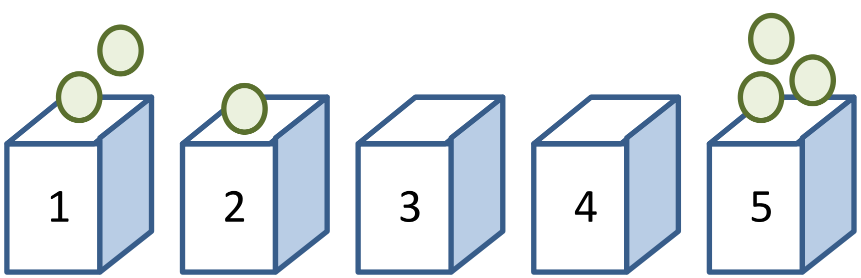


- A particular r-combination is equivalent to throwing a total of r balls into these boxes



With Unlimited Repetitions

- To represent one of the r-combination, we may use a list of n – 1 bars and r stars, where
  - ➢ the bars are used to mark off n different boxes
  - ➢ the stars are used to indicate how many balls in each box
- For instance, suppose n = 5, r = 6



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With Unlimited Repetitions

- Using the bars-and-stars representation, we see that
  - ➢ each r-combination corresponds to a unique representation (with n – 1 bars and r stars), and
  - ➢ each representation (with n – 1 bars and r stars) corresponds to a unique r-combination

➔ # of r-combinations = # of representations

= C(n – 1 + r, r)

Examples

- Ex 1 : Suppose that a cookie shop has four different kinds of cookies. How many different ways can 6 cookies be chosen ?
- Ex 2 : How many solutions does the equation

  $$x + y + z = 11$$

  have, if x, y, z are non-negative integers ?
- Ex 3 : What if x, y, z are positive integers in Ex 2 ?

Interesting Identities

Pascal's Identity :

$$C(n, r) = C(n-1, r) + C(n-1, r-1)$$

- Proof (by combinatorial arguments):

To select r of n objects, there are in two cases :

1. Get the first object, and then get $r-1$ objects from the remaining $n-1$ objects ;
2. Do not get the first object, and get r objects from the remaining $n-1$ objects

➔ In total, $C(n-1, r-1) + C(n-1, r)$ ways

Interesting Identities

Binomial Theorem :

$$(x + y)^n = \sum_{r=0}^{n} C(n, r)\, x^{n-r}\, y^r$$

- Proof (by combinatorial arguments):

  The terms in $(x + y)^n$ must be of the form $x^{n-r} y^r$. To obtain the term $x^{n-r} y^r$, x is chosen n – r times from the n occurrences of (x + y) in the product, so that y will be automatically chosen r times

  ➔ the number of ways is exactly C(n, r)

Examples

- Ex 1 : What is the expansion of $(x + y)^4$?
- Ex 2 : What is the coefficient of x^{12} in $(2x - 3y)^{25}$?
- Ex 3 : What is the value of $\sum_{r=0}^{n} C(n, r)$?
- Ex 4 : What is the value of $\sum_{r=0}^{n} (-1)^r C(n, r)$?
- Ex 5 : What is the value of $\sum_{r=0}^{n} 2^r C(n, r)$?

Interesting Identities

Vandermonde's Identity :

$$C(m + n, r) = \sum_{k=0}^{r} C(m, r-k)\, C(n, k)$$

- Proof (by combinatorial arguments):

  To select r items from m + n distinct objects, we may assume that among these objects, m are white and n are black. The selection may start by selecting k black objects, and then the remaining from white objects. As k can vary from 0 to r, this gives the result.

Example

- Can you simplify $\sum_{k=0}^{n} C(n, k)^2$?

- Answer :

  Observe that

$$\sum_{k=0}^{n} C(n, k)^2 = \sum_{k=0}^{n} C(n, n-k)\, C(n, k)$$

By setting m = n and r = n in Vandermonde's identity, we get the desired value as C(2n, n)