CS 2336
Discrete Mathematics

Lecture 1

Logic: Propositional Logic

Outline

- Proposition
- Propositional Logic
- Applications
- Propositional Equivalences

Proposition

- A declarative sentence is a sentence that declares a fact
  - 1 + 1 = 2. [declarative]
  - 2 + 2 = 3. [declarative]
  - What time is it? [non-declarative]
  - Read this carefully. [non-declarative]

Proposition

- A proposition is a declarative sentence that is either true or false, but not both
  - 1 + 1 = 2. [proposition, true]
  - 2 + 2 = 3. [proposition, false]
  - x + 1 = 2. [non-proposition]
  - You will pass in this course. [??]

Proposition

- We can use variables to denote propositions
  - p: "1 + 1 = 2."
  - q: "You get an A in this course."

- Truth Value
  - If a proposition is true, we say its truth value is true, and is denoted by T
  - Else, its truth value is false, and is denoted by F

Logical Operators

- We can create new proposition from the existing ones, by using logical operators

1. Negation

   Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement

   "It is not the case that p."

   The truth value of $\neg p$ is the opposite of the truth value of p.

Test Your Understanding

- What is the negation of the following propositions? Express them in simple English.
  - Simon's PC runs Windows XP.
  - Chris's PC has at least 32GB of memory.

Logical Operators

2. Conjunction

   Let p and q be propositions.

   The conjunction of p and q,

   denoted by $p \wedge q$, is the proposition

   "p and q."

   The truth value of $p \wedge q$ is true if both p and q are true. Otherwise, it is false.

Test Your Understanding

- What is the conjunction of the following propositions? Express them in simple English.

  – Kai is the lecturer of this course.

  – This course is held on Monday and Friday.

Logical Operators

3. Disjunction

Let p and q be propositions.

The disjunction of p and q,

denoted by $p \vee q$, is the proposition

"p or q."

The truth value of $p \vee q$ is false if both p and q are false. Otherwise, it is true.

Test Your Understanding

- What is the disjunction of the following propositions? Express them in simple English.

  – Students from NTHU can take this class.

  – Students from NCTU can take this class.

Logical Operators

4. Exclusive Or

Let p and q be propositions.

The exclusive or of p and q,

denoted by $p \oplus q$, is the proposition

"either p or q, but not both."

The truth value of $p \oplus q$ is true if exactly one of p and q is true. Else, it is false.

Truth Table

- A convenient way to see the effect of the logical operators is by using a truth table
- The truth table for negation of p is as follows:

p	$\neg p$
T	F
F	T

Truth Table

- The truth table for p $\wedge$ q is as follows:

p	q	p $\wedge$ q
T	T	T
T	F	F
F	T	F
F	F	F

- How about p $\vee$ q and p $\oplus$ q?

Logical Operators

5. Conditional Statement

   Let p and q be propositions.

   The conditional statement $p \rightarrow q$, is the proposition

   "if p, then q."

   The truth value of $p \rightarrow q$ is false if p is true and q is false. Else, it is true.

More on Conditional Statement

- In the proposition $p \to q$,
  - p is called the hypothesis, or the premise
  - q is called the conclusion
- Many arguments in mathematical reasoning involve conditional statements, and there are many equivalent ways to express $p \to q$

  "p implies q" "p only if q" "q if p" "q follows from p"

  "p is sufficient for q" "q is necessary for p"

  "a sufficient condition for q is p"

  "a necessary condition for p is q"

Truth Table

- The truth table for $p \rightarrow q$ is as follows:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- Note: It is not very natural, from English point of view

Test Your Understanding

- Let p and q be the following propositions:
  - p: “You get an A in the course”
  - q: “You pass the course”

- Express the following propositions as conditional statements in terms of p and q:
  - You get an A in the course only if you pass the course
  - You pass the course only if you get an A in the course
  - You get an A in the course implies you pass the course
  - You pass the course implies you get an A in the course

More on Conditional Statement

- Three special propositions are related to the conditional statement $p \to q$
  1. The converse of $p \to q$: $q \to p$
  2. The contrapositive of $p \to q$: $\neg q \to \neg p$
  3. The inverse of $p \to q$: $\neg p \to \neg q$

- What are the converse, contrapositive, and inverse of the following conditional statement?

  "If it rains, Kai orders pizza as his lunch"

Logical Operators

6. Bi-conditional Statement

   Let p and q be propositions.

   The bi-conditional statement $p \leftrightarrow q$, is the proposition

   "p if and only if q."

   The truth value of $p \leftrightarrow q$ is true if p and q have the same truth value. Else, false.

Truth Table

- The truth table for $p \leftrightarrow q$ is as follows:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Any relationship between $p \leftrightarrow q$ and $p \oplus q$?

Compound Propositions

- Using the logical operators, we can build up complicated compound propositions that involves any number of propositions

- Again, we can use truth table to see the truth values of a compound proposition, under all possible combinations of the truth values of the basic simple propositions

Example

- The truth table for the compound proposition

$$(p \lor \neg q) \rightarrow (p \land q)$$

is as follows:

p	q	$\neg q$	$p \lor \neg q$	$p \land q$	$(p \lor \neg q) \rightarrow (p \land q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

- In general, we use parentheses to specify the order in which the logical operators are used
  - However, as in mathematical expressions, we want to reduce the number of parentheses

    $(1 \times 2) + ((-3) \times 4)$ ➔ $1 \times 2 + -3 \times 4$
  - If parentheses are omitted, a general rule is:

    Negation $\neg$ is applied first (similar to - in -3);

    Conjunction $\wedge$ comes next (similar to $\times$)

    Disjunction $\vee$ comes last (similar to +)
  - Rule of Thumb: When in doubt, use parentheses!

Applications

- Translating English Sentences

  How to translate the following sentences into a logical expression?

1. "You have a CS email account only if you are a CS major or you take a CS course."

2. "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

Applications

- Boolean Search

  In searching a collection of data (e.g., files, web pages, book records), we may issue a query with search terms connected by AND, OR, NOT

  E.g., (Tsing AND Hua) OR (Tsinghua)

  SHE AND Band AND Taiwan

  CCU AND (NOT NCCU) AND University

Applications

- Logic Puzzles

In an island, there are two types of people:

one always tell the truth, and one always lie.

You encounter two people A and B.

A says: "B always tells the truth."

B says: "We are of opposite types."

What types of people are A and B?

Applications

- Logic Puzzles

  There are 10 kids, and the teacher gives each one a hat, either white or black. Each kid sees the hats of the others, but not his or hers. The teacher tells the kids that there is at least one white and one black, and asks the following question again and again:

  "Do you know the color of your hat?"

  At the third time, someone finally answers Yes, and says that his hat is black.

  Assume all kids are smart, and answer each question instantly. How many kids are wearing black hats?

Propositional Equivalences

- An important type of mathematical arguments is the replacement of a statement with another one with the same truth value

  Example:

  "n is an even integer"

  can be replaced by

  "n = 2k, for some integer k"

Propositional Equivalences

- Because of the above, methods that produce propositions with the same truth value as a given compound proposition are used extensively in mathematical proofs

- Next, we give some ways to show that two propositions always have the same truth value
  - In this case, they are logically equivalent

Logical Equivalences

- First, we define two special terms:

1. If a compound proposition r is always true, no matter what the truth values of the basic propositions that occur in it, we say r is a tautology

   E.g., $p \vee \neg p$ is a tautology

2. If a compound proposition r is always false, no matter what the truth values of the basic propositions that occur in it, we say r is a contradiction

   E.g., $p \wedge \neg p$ is a contradiction

Logical Equivalences

- Logical Equivalent

  Two propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.

  This enforces that the truth value of p and the truth value of q must always be the same.

  If p and q are logically equivalent, we denote the fact by

$$p \equiv q$$

Logical Equivalences

- One way to show two propositions are logically equivalent is by using a truth table

Ex: Show that $p \rightarrow q$ and $\neg p \vee q$ are equivalent

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Logical Equivalences

- De Morgan's Laws
  1. $\neg (p \vee q) \equiv \neg p \wedge \neg q$
  2. $\neg (p \wedge q) \equiv \neg p \vee \neg q$

- The above two laws can easily be shown by truth table method
- What is the negation of the following?

  "Kai will have pizza or sushi for dinner tonight"

Logical Equivalences

- Important Equivalences

1. Identity Laws

   $p \wedge T_0 \equiv p$ $\quad$ $p \vee F_0 \equiv p$

2. Domination Laws (p is gone)

   $p \wedge F_0 \equiv F_0$ $\quad$ $p \vee T_0 \equiv T_0$

3. Idempotent Laws (okay to apply many times)

   $p \wedge p \equiv p$ $\quad$ $p \vee p \equiv p$

4. Double Negation Law

   $\neg (\neg p) \equiv p$

In the above, T_0 is any tautology, while F_0 is any contradiction.

Logical Equivalences

- Important Equivalences

5. Commutative Laws

$$p \wedge q \equiv q \wedge p \qquad p \vee q \equiv q \vee p$$

6. Associative Laws (parentheses may be omitted)

$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$

7. Distributive Laws (similar to + and × in math expression)

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Logical Equivalences

- Important Equivalences

8. De Morgan's Laws

$$\neg (p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg (p \wedge q) \equiv \neg p \vee \neg q$$

9. Absorption Laws (a proposition may be omitted)

$$p \wedge (p \vee q) \equiv p$$

$$p \vee (p \wedge q) \equiv p$$

10. Negation Laws

$p \wedge \neg p \equiv F_0$ $\qquad$ $p \vee \neg p \equiv T_0$

Logical Equivalences

- Apart from using a truth table, we can show two propositions are equivalent by the logical equivalences that have been established

  Example:

  Show that $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are logically equivalent.

Example (Solution)

$p \rightarrow q \equiv \neg p \vee q$	By Example on Page 33
$\equiv q \vee \neg p$	By Commutative Law
$\equiv \neg(\neg q) \vee \neg p$	By Double Negation Law
$\equiv \neg q \rightarrow \neg p$	By Example on Page 33

Example 2

- Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.
- Solution:

$(p \wedge q) \rightarrow (p \vee q)$

$\equiv \neg (p \wedge q) \vee (p \vee q)$ By Example on Page 33

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ By De Morgan's Law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ By Associative Law and Commutative Law

$\equiv T_0 \vee T_0$ By Commutative Law and Negation Law

$\equiv T_0$ By Domination Law