CS 5319
Advanced Discrete Structure

Lecture 3:

Generating Functions I

Outline

- Introduction
- Generating Functions for

  (1) Combinations

  (2) Permutations
- Distribution of Objects
- More Applications

This Lecture

Introduction

Introduction

- Suppose we have 3 objects: a, b, c
- There are 3 ways to select 1 object from them
  - We may describe this by:

$$a + b + c$$

- There are 3 ways to select 2 objects from them
  - We may describe this by:

$$ab + bc + ca$$

Introduction

- There is only one way to select all objects

  We may describe this by:

$$a\ b\ c$$

- What is so special about these terms ?

Introduction

- Let us consider the polynomial

$$(1 + ax) (1 + bx) (1 + cx)$$

- After expansion, we get

$$1 + (a + b + c)\, x + (ab + bc + ca)\, x^2 + abc\, x^3$$

- Are the coefficients familiar ?

Introduction

- We can interpret the polynomial by

Rules of Sum and Product

- The sum ($1 + ax$) means that for object a, the ways of selection include :

“not select a” OR “select a”

- We may use the term ($1 + a$) instead, but we see that the variable x is useful because it can indicate the case where one object is selected

Introduction

- The product

$$(1 + ax)(1 + bx)(1 + cx)$$

means for objects a, b, c

the ways of selection are :

"not select a" or "select a" AND

"not select b" or "select b" AND

"not select c" or "select c"

Introduction

- Consequently :

Powers of x indicates
how many objects are selected

Coefficients of x^k indicates
the different ways to select k objects

- What is the meaning of the constant term in the polynomial ?

Generating Functions

- The above example motivates us to use a polynomial to represent a sequence of terms (or a sequence of numbers)
- Ex: We may represent the sequence of numbers (1, 3, 6, 10, 15, …) by

$$1 + 3x + 6x^2 + 10x^3 + 15x^4 + \dots$$

This polynomial is called the generating function of the sequence

Generating Functions

- The roles of x, x^2, x^3, … are just indicators
- We may as well use another set of indicators
- Ex: We may represent the sequence of numbers (1, 3, 6, 10, …) by

$$1 + 3 \cos x + 6 \cos 2x + 10 \cos 3x + \ldots$$

or by

$$1 + 3x^{\underline{1}} + 6x^{\underline{2}} + 10x^{\underline{3}} + \ldots$$

We use $x^{\underline{k}}$ to denote the falling function $x(x-1)(x-2)\ldots(x-k+1)$

Generating Functions

- However, some indicators are not preferred
- Ex: One may use

$$1,\ 1+x,\ 1-x,\ 1+x^2,\ 1-x^2,\ \ldots$$

as indicators

Why is it not good ?

Let us consider the sequence

$$(2, 0, 0, 0, 0, \ldots) \quad \text{and} \quad (0, 1, 1, 0, 0, \ldots)$$

Generating Functions

- In general, the most useful set of indicators is

$$1,\ x,\ x^2,\ x^3,\ \ldots$$

so that a sequence $(a_0, a_1, a_2, \ldots, a_r, \ldots)$ is represented by

$$F(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots$$

- Such generating functions will be our focus
- Next, we shall see how to apply generating functions to solve combinatorial problems

Generating Functions for Combinations

GF for Combinations

- We have seen that

$$(1 + ax)(1 + bx)(1 + cx)$$

is the generating function of the different ways to select objects a, b, c

- Instead of different ways of selecting a certain # of objects, we may be interested only in number of ways of selecting a certain # of objects

GF for Combinations

- By setting $a = b = c = 1$, we have

$$(1 + x)(1 + x)(1 + x) = 1 + 3x + 3x^2 + x^3$$

- The coefficient of x^r is exactly the number of ways to select r objects
- This generating function gives the number of combinations

➔ We call this an enumerator

GF for Combinations

- We can extend the idea to find the number of combinations of n objects, by the enumerator

$$(1 + x)^n = 1 + C(n,1)\,x + C(n,2)\,x^2 + C(n,3)\,x^3 + \dots + C(n, n-1)\,x^{n-1} + x^n$$

- Again, the coefficient of x^r is exactly the number of ways to select r objects. Why ?

Reason: Each x^r term is obtained by selecting r x's and $n-r$ 1's among the n factors of $(1 + x)$

GF for Combinations

- Example Applications :
- Show that

$$C(n,0) + C(n,1) + C(n,2) + \dots + C(n,n) = 2^n$$

- Show that

$$C(n,0) - C(n,1) + C(n,2) - \dots + (-1)^n C(n,n) = 0$$

GF for Combinations

- Show that

$$C(n,0)^2 + C(n,1)^2 + C(n,2)^2 + \ldots + C(n,n)^2$$
$$= C(2n, n)$$

- Method 1: Consider the constant term of $(1 + x)^n(1 + x^{-1})^n$
- Method 2: Use combinatorial arguments, and rewrite $C(n, x)^2$ as $C(n, x)C(n, n-x)$

GF for Combinations

- Show that

$$C(n,1) + 2C(n,2) + 3C(n,3) + \ldots + n\,C(n,n)$$
$$= n\,2^{n-1}$$

- Hint: Differentiation on $(1 + x)^n$

GF for Combinations

- Find the coefficient of x^{23} in

$$(1 + x^5 + x^9)^{100}$$

- Hint: How can we obtain the term x^{23}?

GF for Combinations

- Show that the coefficient of x^r in

$$(1 - 4x)^{-1/2}$$

is $C(2r, r)$

- In other words, $(1 - 4x)^{-1/2}$ is the generating function of

$$C(0,0), C(2,1), C(4,2), C(6,3), \ldots$$

GF for Combinations

- With the previous result, we can show that

$$\sum_{r=0}^{t} C(2r, r)\, C(2t-2r, t-r) = 4^t$$

- Hint: Can we obtain the sum on the left side as the coefficient of some function (or the product of some functions) ? See Page 19, Method 1

GF for Combinations

- What is the meaning of this ?

$$(1 + ax + a^2x^2)(1 + bx)(1 + cx)$$

This represents the case where object *a* can be selected twice

- What is the meaning of this ?

$$(1 + ax)(1 + a^2x^2)(1 + bx)(1 + cx)$$

GF for Combinations

- Ex: Suppose we have *p* kinds of objects, each with two pieces, and *q* additional kinds of objects, each with one piece

  Argue that the number of ways to select *r* pieces of objects is :

$$\sum_{i=0}^{\lfloor r/2 \rfloor} C(p, i)\, C(p + q - i, r - 2i)$$

GF for Combinations

- What is the meaning of this ?

$$(1 + x + x^2 + x^3 + \cdots + x^k + \cdots)^n$$

This represents the case where we select from *n* objects, and each object has unlimited supply

- What is the coefficient of x^r ?

GF for Combinations

- Since

$$(1 + x + x^2 + x^3 + \cdots + x^k + \cdots)^n$$

$$= (1 - x)^{-n}$$

$$= 1 + C(-n,1)(-x) + C(-n,2)(-x)^2 + \cdots$$

- Thus the coefficient of x^r is :

$$C(-n, r)(-1)^r = (-n)^{\underline{r}}(-1)^r / r!$$

$$= |C(-n, r)| = C(n + r - 1, r)$$

GF for Combinations

- What is the meaning of this ?

$$(x^q + x^{q+1} + x^{q+2} + \cdots + x^{q+z-1})^n$$

This represents the case where we select from n objects, and each object is chosen with at least q and at most $q + z - 1$ copies

- What is the coefficient of x^r ?

GF for Combinations

- Since

$$(x^q + x^{q+1} + x^{q+2} + \cdots + x^{q+z-1})^n$$

$$= \left(x^q (1 + x^1 + x^2 + \cdots + x^{z-1}) \right)^n$$

$$= \left(x^q (1 - x^z) / (1 - x) \right)^n$$

$$= x^{qn} \left((1 - x^z) / (1 - x) \right)^n$$

➔ The desired coefficient of x^r is equal to the coefficient of x^{r-qn} in $\left((1 - x^z) / (1 - x)\right)^n$

GF for Combinations

- Ex: Suppose we have four persons, each rolling a die once.

  How many ways can the total score be 17?
- Set $r = 17$, $n = 4$, $q = 1$, $z = 6$
- The desired answer is the coefficient of x^{13} in

$$\left((1 - x^6) / (1 - x)\right)^4$$

GF for Combinations

- Since

$$(1-x^6)^4 = 1 - 4x^6 + 6x^{12} - 4x^{18} + x^{24}$$

$$(1-x)^{-4} = 1 + |C(-4,1)|\,x + |C(-4,2)|\,x^2 + |C(-4,3)|\,x^3 + \ldots$$

➔ The coefficient of x^{13} in $\left((1-x^6)/(1-x)\right)^4$ is equal to :

$$|C(-4,13)| - 4 \times |C(-4,7)| + 6 \times |C(-4,1)| = 104$$