CS 5319
Advanced Discrete Structure

Permutations and Combinations I

Outline

- Notation
- Rules of Sum and Product
- Permutations
- Combinations
- Distribution of Objects

* Stirling's Formula

This Lecture

Notation

- Selection and arrangement of objects appear in many places
  → we often want to compute # of ways to select or arrange the objects
- Ex:

1. How many ways to select 2 people from 5 candidates ?
2. How many ways to arrange 5 books on the bookshelf ?

Notation

- In most textbooks, we use the word

  combination ⇔ selection

> Definition: An ***r*-combination of *n* objects** is an unordered selection of *r* of these objects

- Ex: { *c*, *d* } is a 2-combination of the 5 objects { *a*, *b*, *c*, *d*, *e* }

Notation

- In most textbooks, we use the word

  permutation ⇔ arrangement

Definition: An ***r*-permutation of *n* objects** is an ordered arrangement of *r* of these objects

- Ex: *cbade* is a 5-permutation of the 5 objects { *a*, *b*, *c*, *d*, *e* }

Notation

- Further, we use the following notation:

The notation ***C*(*n*,*r*)** denotes the number of *r*-combination of *n* distinct objects

The notation ***P*(*n*,*r*)** denotes the number of *r*-permutation of *n* distinct objects

- What are the values of $C(n,n)$, $C(n,1)$, $C(3,2)$, and $P(3,2)$?

Rules of Sum and Product

- Suppose we have

5 Roman letters	A, B, C, D, E
and 3 Greek letters	α, β, γ

- How many ways to select two letters, one from each group ?
- How many ways to select one letter that is either a Roman or a Greek letter ?

Rules of Sum and Product

- In general, if

  one event can occur in m ways and
  another event can occur in n ways,

Rule of Product: There are $m \times n$ ways that these two events can occur together

Rule of Sum: There are $m + n$ ways that one of these two events can occur

Rules of Sum and Product

Ex: Suppose there are

5 Chinese books, 7 English books, and

10 French books

- How many ways to choose 2 books of different languages from them?

- Ans: $5 \times 7 + 5 \times 10 + 7 \times 10 = 155$

Rules of Sum and Product

Ex:

Why are the following formulas correct?

1. $P(n,r) = P(r,r) \times C(n,r)$
2. $P(n,n) = P(n,r) \times P(n-r,n-r)$
3. $C(n,r) = C(n-1,r-1) + C(n-1,r)$

Permutations

Permutations

- We can show that

$$P(n,r) = n \times (n-1) \times \ldots \times (n-r+1)$$
$$= n! \,/\, (n-r)!$$

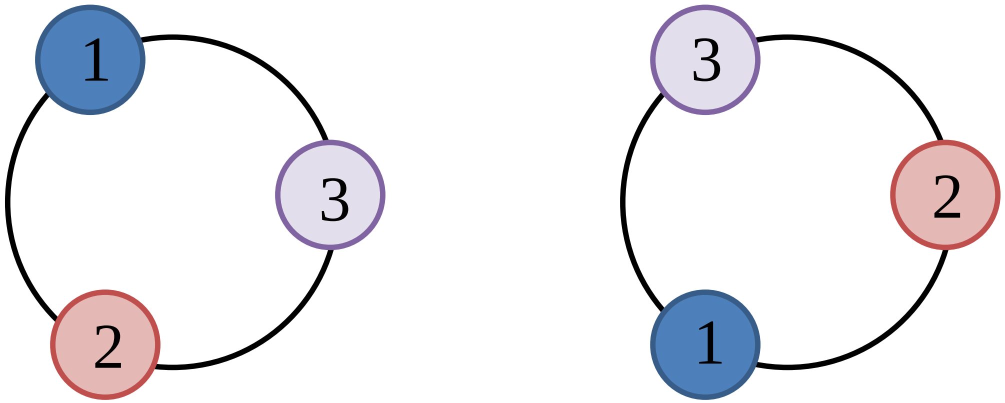
Proof : $P(n,r)$ = # ways to get r of n objects in some order.

There are n ways to get the first object, $n-1$ ways to get the second object, ... , $n-r+1$ ways to get the last object.

➔ Result follows from rule of product.

Permutations

Ex: How many ways can *n* people stand to form a ring ?



The above are considered to be the same

(as relative order is the same)

Permutations

- Suppose we have n objects which are not all distinct, where

  q_1 objects are of the first kind,

  q_2 objects are of the second kind,

  …

  q_t objects are of the t th kind.

➔ # of n-permutation of these objects is :

$$\frac{n!}{q_1!\, q_2! \ldots q_t!}$$

Permutations

Ex:

Suppose we have 5 dashes and 8 dots

➔ 13! / (5!8!) ways to arrange them

- If we can only use 7 symbols from them, how many different arrangements?

Ans: 7! / (5!2!) + 7! / (4!3!) + 7! / (3!4!) +
7! / (2!5!) + 7! / (1!6!) + 7! / 7! = 120

Permutations

Ex: How to show that

$(k!)!$ is divisible by $k!^{(k-1)!}$?

- Consider the permutation of $k!$ objects, where

  k are of the first kind,

  k are of the second kind,

  …

  k are of the $(k-1)!$ th kind.

Permutations

- Suppose we have *n* distinct objects, each with unlimited supply
- The # of ways to arrange *r* objects from them is:

$$n^r$$

Permutations

Ex: Consider the numbers between 1 and 10^{10}.

- How many of them contain the digit 1?
- How many of them do not ?

Ans: $9^{10} - 1$ of them do not, the others do

Permutations

Ex: Consider all *n*-digit binary strings.

- How many contain even number of 0's ?

Ans: Half of them (by symmetry)

Ex: Consider all *n*-digit quaternary strings.

- How many contain even number of 0's ?

Ans: $2^n + (4^n - 2^n) / 2$ (how to get this?)

Combinations

Combinations

- Recall that

$$P(n,r) \;=\; P(r,r) \times C(n,r)$$

Thus we have:

$$\begin{aligned} C(n,r) &= P(n,r) \,/\, P(r,r) \\ &= n! \,/\, [\,(n-r)!\, r!\,] \end{aligned}$$

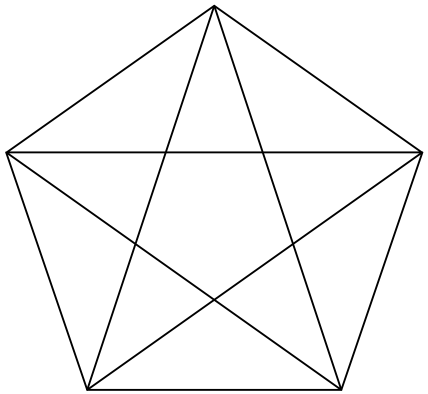
- Immediately, we also have

$$C(n,r) \;=\; C(n,n-r)$$

Combinations

Ex: Consider a decagon (10-sided) where no three diagonals meet at a point.

- How many line segments are the diagonals divided by their intersections ?



In case of a pentagon,
there will be 15 line segments

Combinations

Ex:

Five pirates have discovered a treasure box.

They decide to keep that in a locked room so that all the locks can be opened if and only if 3 or more pirates are present

- How can they do so? How many locks needed?

(Each pirate can possess keys to different locks)

Combinations

Ex:

In how many ways can we select three numbers from 1,2, …, 300 such that their sum is divisible by 3 ?

- When the sum of three numbers is divisible by 3, what special property do they have?

- Ans: $C(100,3) + C(100,3) + C(100,3) + 100^3$

Combinations

- Suppose we have *n* distinct kinds of objects, each with unlimited supply
- The # of ways to select *r* objects from them is:

$$C(n+r-1, r)$$

- How to prove it ?

Combinations

Ex:

When three indistinguishable dice are thrown, how many outcomes are there ?

- Ans: $C(6+3-1,3) = 56$

Combinations

Ex:

Out of a number of \$100, \$200, \$500, \$1000 bills, how many ways can six bills be selected ?

- Ans: $C(4+6-1,6) = 84$

Combinations

- Suppose we have n objects which are not all distinct, where

  q_1 objects are of the first kind,

  q_2 objects are of the second kind,

  …

  q_t objects are of the t th kind.

➔ # of ways to select one or more of these objects from them is :

$$(q_1+1)(q_2+1) \ldots (q_t+1) - 1$$

Combinations

Ex:

How many divisors does 1400 have ?

- Ans:

Since $1400 = 2^3 \times 5^2 \times 7$,

the number of divisors of 1400 is

$$(3+1)\,(2+1)(1+1) = 24$$

Combinations

Ex:

For *n* given weights, what is the greatest number of different amounts that can be made up by the combinations of these weights?

To weigh things with integral weight between 1 and 100, how many weights do we need ?

Combinations

Ex:

What is the greatest number of different amounts that can be weighed using n weights and a balance ?

To weigh things with integral weight between 1 and 100, how many weights do we need ?