CS5319 Advanced Discrete Structure

Homework 4

Due: 1:10 pm, December 2, 2010 (before class)

1. Fermat once conjectured that for $n \geq 0$, all numbers $F_n = 2^{2^n} + 1$ are primes. Indeed, the numbers
$$F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$$
are all primes.

   This conjecture was later disproved by Euler in 1732, who showed that $F_5 = 4294967297 = 641 \times 6700417$. In spite of this, Fermat numbers have many interesting properties. In this question, we shall base on it to give an alternative proof (due to Goldbach) that there are infinitely many primes.

   (a) Show that for all $n \geq 1$, $F_n = F_0 \times F_1 \times F_2 \times \cdots \times F_{n-1} + 2$.

   (b) Using the result of (a), argue that Fermat numbers are pairwise relatively prime.

   (c) Show that if we pick one prime factor from each Fermat number, they must be all distinct.

   (d) Using the result of (c), conclude that there are infinitely many primes.

2. Let $\triangle$ denote an equilateral triangle with the length of each side equal to 2 units. Show that by placing 5 points inside $\triangle$, we can always find two points whose distance is at most 1 unit. (*Hint:* Pigeonhole's principle.)

3. Consider a $2m \times 2n$ grids, so that each grid point is either colored red or blue, with number of red points is equal to number of blue points, for any row and any column. When two adjacent points (up, down, left, right) have the same color c, we join them by a line of color c (See Figure 1 for an example). Show that no matter how the original points are colored, the number of red lines must be equal to the number of blue lines.

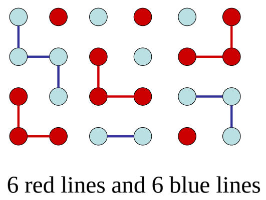

Figure 1: Number of red lines and blue lines are the same.

4. Give a sequence of n integers $a_1, a_2, a_3, ..., a_n$, show that there exists a contiguous subsequence whose sum is divisible by n.

5. Consider a game played on an infinite checkerboard where each square below a certain horizontal line is occupied by a piece. Each move can jump a piece horizontally or vertically over another piece on to an empty square, where the jumped-over piece is then removed. Our target is to advance a piece as far away above the horizontal line as possible.

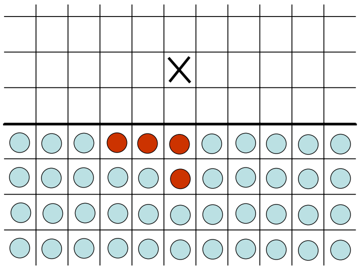

Figure 2: Advancing a piece above the horizontal line.

For instance in Figure 2, we can make use of the red pieces to advance a piece two squares above the horizontal line (to the target square marked by a cross). Surprisingly, it was shown that there is a limit to which we can advance a piece.

Suppose each square is associated with a value. In particular, the target square has a value 1, and each square with distance n from the target square has value x^n where $x = 0.9$. The *score* of the checkerboard is the total value of the squares occupied by all the pieces in the checkerboard.

(a) What is the score initially in terms of n?

(b) After a move, two pieces in the board will be replaced by a third one, so that there will be a change in the score. Show that when $x = 0.9$, the score will be decreased after each move.

(c) Based on the result of (b), give an upper bound on the value of n. (*Hint:* Recall that the target square has value 1.)

(d) By tuning the value of x, we can actually show $n < 5$. How?