Gamma Function and the Volumes of High Dimensional Spheres

1. Define $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$ and let $\gamma = \int_0^\infty e^{-x^2}dx$. Then

(a) $\Gamma(x) = \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

(b) Show that $\Gamma(\frac{1}{2}) = 2\gamma = \sqrt{\pi}$

(c) $\Gamma(x+1) = x\Gamma(x),\ for\ x > 0,\ \Gamma(n) = (n-1)!$ if $n \in N$.

(d) The volume of a $d - dimensional$ unit sphere is $\pi^{d/2}/\Gamma(\frac{d}{2}+1)$.

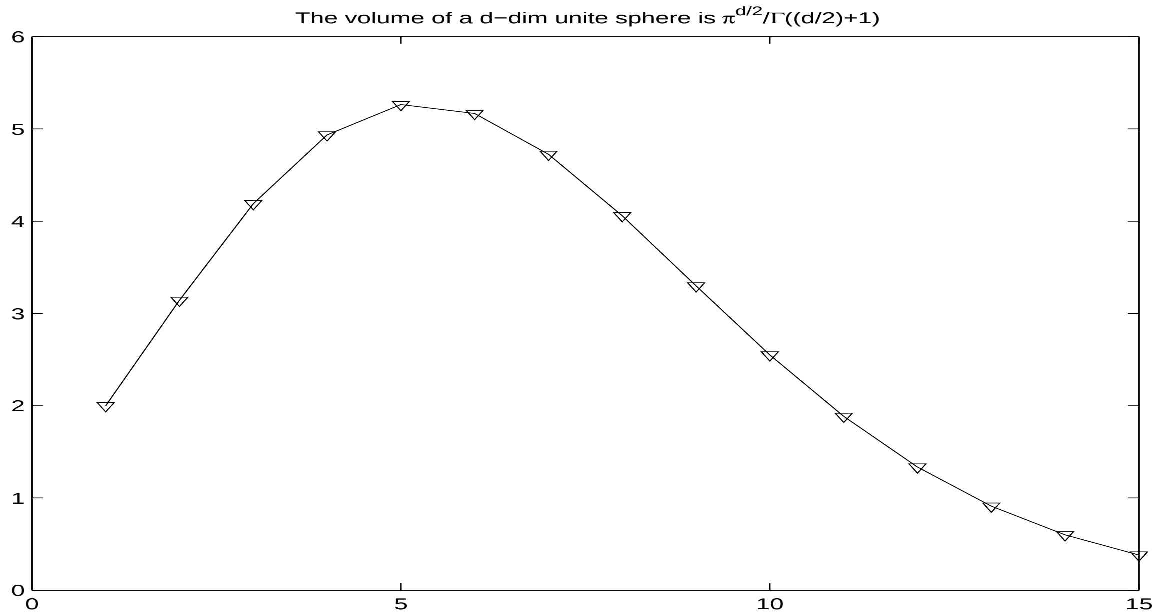

Figure 1: The Volume of a High Dimensional Sphere.

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V(1)=2;      V(2)=pi;    V(3)=4*pi/3;
V(4)=pi*pi/2;            V(5)=8*pi*pi/15;       V(6)=pi*pi*pi/6;
V(7)=16*pi*pi*pi/105;    V(8)=(pi)^4/24;        V(9)=32*(pi)^4/945;
V(10)=(pi)^5/120;        V(11)=64*(pi)^5/10395; V(12)=(pi)^6/720;
V(13)=128*(pi)^6/135135; V(14)=(pi)^7/5040;     V(15)=256*(pi)^7/2027025;
D=1:15;
plot(D,V,'b-v')
title('The volume of a d-dim unit sphere is \pi^{d/2}/\Gamma((d/2)+1)')
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The Derivation of Volumn for an n-Dimensional Sphere

□ For $n = 2$,

$$\begin{aligned} x_1 &= r\cos\theta, \quad 0 \le \theta \le 2\pi \\ x_2 &= r\sin\theta, \quad 0 \le r \le R \end{aligned} \quad , \quad J_2 = \frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} = r$$

The volumn is computed by

$$V_2 = \int_0^R \int_0^{2\pi} J_2 dr d\theta = \int_0^R \int_0^{2\pi} r dr d\theta = \pi R^2$$

□ For $n = 3$,

$$\begin{aligned} x_1 &= r\cos\theta_1\cos\theta_2, \quad 0 \le \theta_2 \le 2\pi \\ x_2 &= r\cos\theta_1\sin\theta_2, \quad -\tfrac{\pi}{2} \le \theta_1 \le \tfrac{\pi}{2} \\ x_3 &= r\sin\theta_1, \quad 0 \le r \le R \end{aligned} \quad , \quad J_3 = \frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} & \frac{\partial x_3}{\partial r} \\ \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_1} \\ \frac{\partial x_1}{\partial \theta_2} & \frac{\partial x_2}{\partial \theta_2} & \frac{\partial x_3}{\partial \theta_2} \end{vmatrix} = r^2\cos\theta_1$$

The volumn is computed by

$$V_3 = \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} J_3 dr d\theta_1 d\theta_2 = \int_0^R \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2\cos\theta_1 dr d\theta_1 d\theta_2 = \frac{4\pi R^3}{3}$$

□ For $n \geq 4$,

$$\begin{aligned}
&x_1 = r\cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{n-2}\cos\theta_{n-1}, && 0 \leq \theta_{n-1} \leq 2\pi \\
&x_2 = r\cos\theta_1\cos\theta_2\cdots\cdots\cos\theta_{n-2}\sin\theta_{n-1}, && -\tfrac{\pi}{2} \leq \theta_{n-2} \leq \tfrac{\pi}{2} \\
&x_3 = r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-3}\sin\theta_{n-2}, && -\tfrac{\pi}{2} \leq \theta_{n-3} \leq \tfrac{\pi}{2} \\
&\vdots && \vdots \\
&x_j = r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-j}\sin\theta_{n-j+1}, && -\tfrac{\pi}{2} \leq \theta_{n-j} \leq \tfrac{\pi}{2} \\
&\vdots && \vdots \\
&x_{n-1} = r\cos\theta_1\sin\theta_2, && -\tfrac{\pi}{2} \leq \theta_1 \leq \tfrac{\pi}{2} \\
&x_n = r\sin\theta_1, && 0 \leq r \leq R
\end{aligned}$$

Note that $\sum_{i=1}^{n} x_i^2 = r^2$ and denote $c_i = \cos\theta_i$, $s_i = \sin\theta_i$ for $1 \leq i \leq n-1$. Then the Jacobian $J_n = \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(r, \theta_1, \cdots, \theta_{n-1})}$ is computed as

$$J_n = r^{n-1}\begin{vmatrix}
c_1c_2\cdots c_{n-2}c_{n-1} & c_1c_2\cdots c_{n-2}s_{n-1} & c_1c_2\cdots c_{n-3}s_{n-2} & \cdots & s_1 \\
-s_1c_2\cdots c_{n-2}c_{n-1} & -s_1c_2\cdots c_{n-2}s_{n-1} & -s_1c_2\cdots c_{n-3}s_{n-2} & \cdots & c_1 \\
-c_1s_2\cdots c_{n-2}c_{n-1} & -c_1s_2\cdots c_{n-2}s_{n-1} & -c_1s_2\cdots c_{n-3}s_{n-2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-c_1c_2\cdots s_{n-2}c_{n-1} & -c_1c_2\cdots s_{n-2}s_{n-1} & c_1c_2\cdots c_{n-3}c_{n-2} & \cdots & 0 \\
-c_1c_2\cdots c_{n-2}s_{n-1} & c_1c_2\cdots c_{n-2}c_{n-1} & 0 & \cdots & 0
\end{vmatrix}$$

Then

$$J_n = r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -t_1 & -t_1 & -t_1 & \cdots & \frac{1}{t_1} \\ -t_2 & -t_2 & -t_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n-2} & -t_{n-2} & \frac{1}{t_{n-2}} & \cdots & 0 \\ -t_{n-1} & \frac{1}{t_{n-1}} & 0 & \cdots & 0 \end{vmatrix}, \quad where \;\; t_i = \frac{sin\theta_i}{\cos\theta_i}$$

Subtracting each column from the preceding one and do further simplifications, we obtain

$$\begin{aligned} J_n &= r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \Pi_{j=1}^{n-1} (t_j + \frac{1}{t_j}) \\ &= r^{n-1} c_1^{n-1} c_2^{n-2} \cdots c_{n-1}^1 s_1 s_2 \cdots s_{n-1} \Pi_{j=1}^{n-1} (\frac{1}{s_j \cdot c_j}) \\ &= r^{n-1} c_1^{n-2} c_2^{n-3} \cdots c_{n-3}^2 c_{n-2}^1 \\ &= r^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \cdots \cos^2\theta_{n-3} \cos^1\theta_{n-2} \end{aligned}$$

Therefore, the volumn of an n-dimensional sphere can be calculated by

$$\begin{aligned} V_n &= \int_0^R \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} (J_n) d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1} dr \\ &= \int_0^R \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} [r^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \cdots \cos\theta_{n-2}] d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1} dr \\ &= \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot R^n \end{aligned}$$

Note that the above computations exploit the properties of the following Gamma and Beta functions, and trignometry.

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty e^{-t} t^{\alpha-1} dt \quad for\ \alpha > 0 \\ Beta(\alpha, \beta) &= \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx, \quad where\ \alpha, \beta > 0 \\ Beta(\alpha, \beta) &= \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

- *Relationship Between Gamma and Beta Functions*

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty e^{-u} u^{x-1} du \int_0^\infty e^{-v} v^{y-1} dv \\ &= \int_0^\infty \int_0^\infty e^{-u-v} u^{x-1} v^{y-1} du dv \\ &= \int_{z=0}^\infty \int_{t=0}^1 e^{-z} (zt)^{x-1} [z(1-t)]^{y-1} z dt dz \quad by\ putting\ u = zt,\ v = z(1-t) \\ &= \int_{z=0}^\infty e^{-z} z^{x+y-1} dz \int_{t=0}^1 t^{x-1} (1-t)^{y-1} dt \\ &= \Gamma(x+y) Beta(x, y) \end{aligned}$$

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \ for\ \alpha > 1$$

$$\Gamma(1) = 1, \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\int_0^{2\pi} d\theta_{n-1} = 2\pi, \qquad \int_{-\pi/2}^{\pi/2} \cos\theta_{n-2} d\theta_{n-2} = 2,$$

$$\int_{-\pi/2}^{\pi/2} \cos^2\theta_{n-3} d\theta_{n-3} = \frac{\pi}{2}, \quad \int_{-\pi/2}^{\pi/2} \cos^m\theta d\theta = \int_0^{\pi/2} 2\cos^m\theta d\theta, \quad 3 \le \alpha \le n-2$$

Now

$$\begin{aligned}
\int_0^{\pi/2} 2\cos^m\theta d\theta \quad &= \int_1^0 2x^m \frac{1}{-\sqrt{1-x^2}} dx \quad \textit{by letting } x = \cos\theta \\
= \int_0^1 2x^m (1-x^2)^{-1/2} dx \quad &= \int_0^1 2y^{m/2}(1-y)^{-1/2} \frac{1}{2\sqrt{y}} dy, \quad \textit{where} \;\; x = \sqrt{y} \\
= \int_0^1 y^{\frac{m}{2}-\frac{1}{2}}(1-y)^{-1/2} dy \quad &= \int_0^1 2y^{\frac{m+1}{2}-1}(1-y)^{\frac{1}{2}-1} dy \\
= Beta(\tfrac{m+1}{2}, \tfrac{1}{2}) \quad &= \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{m}{2}+1)}
\end{aligned}$$

Therefore,

$$V_n = \int_0^R \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cdots \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} [r^{n-1}\cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \cdots \cos\theta_{n-2}] d\theta_1 d\theta_2 \cdots d\theta_{n-2} d\theta_{n-1} dr$$

$$= \tfrac{R^n}{n} \cdot (2\pi) \cdot (2) \cdot (\tfrac{\pi}{2}) \cdot \Pi_{m=3}^{n-2} Beta(\frac{m+1}{2}, \; \frac{1}{2}) \; = \; \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} \cdot R^n$$