Exam 2 for EECS3030(02), Spring 2020

10:10-11:50 Weddnesday, June 17, 2020

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(30pts) 1. Fill the following blanks.

(a) Let $\{X_i \sim b(16, 0.75),\ 1 \le i \le 10\}$ be a random sample from a binomial distribution. Define $X = \sum_{i=1}^{10} X_i$, then

the moment-generating function $M_X(t) = \underline{(0.25 + 0.75e^t)^{160}}$, $Var(X) = \underline{30}$

the probability mass function of X, $f_X(x) = \underline{C(160, x)(0.75)^x(0.25)^{160-x},\ 0 \le x \le 160}$

(b) Let $\{Y_i,\ 1 \le i \le 8\}$ be a random sample of Poisson distribution with variance $Var(Y_8) = 2$ and define $Y = \sum_{i=1}^{8} Y_i$. Then

the moment-generating function $M_Y(t) = \underline{e^{16(e^t-1)}}$

the p.m.f. of Y, $f_Y(y) = \underline{\frac{e^{-16}(16)^y}{y!},\ y = 0, 1, \cdots,}$, $Var(Y) = \underline{16}$

(c) Let the joint probability distribution function of the lifetimes of two brands of mobile phones be given by $F(x, y) = (1 - e^{-x})(1 - e^{-y})\ \ if\ x > 0,\ y > 0$. Then the probability that one mobile phone lasts more than three times as long as the other is $\underline{\frac{1}{2}}$

(d) Let the joint probability density function of random variables X and Y be given by $f(x, y) = 8xy,\ \ 0 \le x \le y \le 1$. Then

$f_X(x) = \underline{4x - 4x^3,\ 0 \le x \le 1}$, $E(Y) = \underline{\frac{4}{5}}$, $\rho(X, Y) = \underline{\frac{4}{\sqrt{66}}}$

(e) Let the conditional probability density function of X, given that $Y = y$, be

$$f_{X|Y}(x|y) = \frac{x + y}{1 + y} e^{-x},\ \ 0 < x < \infty,\ 0 < y < \infty.$$

$P(X < 4|Y = 1) = \underline{1 - 3e^{-4}}$

(30pts) 2. Fill the following blanks.

(a) Suppose that X and Y are independent and identically distributed exponential random variables with variance 4. Define $W = \frac{Y}{X+Y}$. Then

the probability density function $f_Y(y) = \underline{\ \frac{1}{2}e^{-y/2},\ y \geq 0,}$

$E(W) = \underline{\ \frac{1}{2}}$, $Var(W) = \underline{\ \frac{1}{12}}$

(b) Let $\{X_i,\ 1 \leq i \leq 12\}$ be a random sample with the probability density function $f_{X_1}(x) = \frac{1}{16}x^2 e^{-x/2},\quad x > 0$. Define $W = \sum_{i=1}^{12} X_i$. Then

the moment-generating function $M_{X_3}(t) = \underline{\ \frac{1}{(1-2t)^3}}$, $Var(X_3) = \underline{\ 12}$

the moment-generating function $M_W(t) = \underline{\ \frac{1}{(1-2t)^{36}}}$

(c) Let $\{X_1, X_2, \cdots, X_9\}$ be a random sample of size n=*9* from the *exponential distribution* with $Var(X_4) = 4$. Define $Y = \sum_{i=1}^{9} X_i$. Then

the moment-generating function of Y, $\phi_Y(t) = \underline{\ \frac{1}{(1-2t)^9}}$,

the p.d.f. of Y, $f_Y(y) = \underline{\ \frac{1}{\Gamma(9)2^9}y^8 e^{-y/2},\ y > 0,}$ $Var(Y) = \underline{\ 36}$

(d) Let $Z \sim N(0, 1)$. Define $Y = 3Z + 2$, then

the p.d.f. of Y, $f_Y(y) = \underline{\ \frac{1}{\sqrt{18\pi}}e^{-(y-2)^2/18},\quad -\infty < y < \infty}$

the moment-generating function of Y, $\phi_Y(t) = \underline{\ e^{2t+\frac{9t^2}{2}}}$, $Var(Y) = \underline{\ 9}$

(e) Let $\{X_i \sim b(12, 0.5),\ 1 \leq i \leq n\}$ be a random sample from a binomial distribution of size n. Define $W_n = (\sum_{i=1}^{n} X_i - 6n)/\sqrt{3n}$.

$E(X_n) = \underline{\ 6}$, $Var(X_n) = \underline{\ 3}$ According to the Central Limit Theorem,

the limiting moment-generating function, $limit_{n\to\infty}M_{W_n}(t) = \underline{\ e^{t^2/2}}$

(10pts) 3. Let $\{X_i \sim N(3,4),\ 1 \le i \le n\}$ be a random sample of a normal distribution. Define $W = \sum_{i=1}^{n} \frac{X_i - 3}{2}$, $Y = \sum_{i=1}^{n} \left(\frac{X_i - 3}{2}\right)^2$.

(a) Write down the probability density function of X_n.

(b) Calculate the moment-generating function of W and its probability density function, $f_W(w)$, respectively.

(c) Calculate the moment-generating function of Y and its probability density function, $f_Y(y)$, respectively.

(d) Calculate $Var(W),\ Var(Y)$, respectively.

(Sa) $f_{X_n}(x) = \frac{1}{\sqrt{8\pi}} e^{-(x-3)^2/8}, \quad -\infty < x < \infty$

(Sb) Let $Z_i = \frac{X_i - 3}{2} \sim N(0,1)$, then $W = \sum_{i=1}^{n} Z_i$. Hence, $M_W(t) = e^{nt^2/2}$, and $f_W(w) = \frac{1}{\sqrt{2n\pi}} e^{-w^2/2n}, \quad -\infty < w < \infty$

(Sc) $Y \sim \chi^2(n)$, then $M_Y(t) = \frac{1}{(1-2t)^{n/2}}$, $f_Y(y) = \frac{1}{\Gamma(\frac{n}{2}) 2^{n/2}} y^{n/2-1} e^{-y/2},\ y > 0.$

(Sd) $Var(W) = n, \quad Var(Y) = 2n.$

(10 pts) 4. The time it takes for a student to finish an aptitude test (in hours) has the probability density function

$$f(x) = 6(x-1)(2-x) \ \ if \ 1 < x < 2, \ \ and \ \ 0 \ \ elsewhere.$$

Approximate the probability that the average length of time it takes for a random sample of 16 students to complete the test in less than 1 hour and 45 minutes (that is, 1.75 hours).

(Hint) You can express your solution in terms of $\Phi(r)$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, and $r \in (-4.99, 4.99)$.

(Solution)

$$E(X) = \int_1^2 x \cdot 6(x-1)2-x)dx = \int_1^2 -6x^3 + 18x^2 - 12xdx = \tfrac{3}{2},$$

$$E(X^2) = \int_1^2 x^2 \cdot 6(x-1)2-x)dx = \int_1^2 -6x^4 + 18x^3 - 12x^2dx = \tfrac{23}{10},$$

$$Var(X) = \tfrac{23}{10} - (\tfrac{3}{2})^2 = \tfrac{1}{20}.$$

Let $\bar{X} = \sum_{i=1}^{16} X_i$, by Central Limit Theorem, $P\left(\frac{\bar{X}-\frac{3}{2}}{\frac{1}{\sqrt{20}} \cdot \frac{1}{\sqrt{16}}} \leq \frac{1.75-\frac{3}{2}}{\frac{1}{8\sqrt{5}}}\right) \approx \Phi(2\sqrt{5}) = \Phi(4.472)$.

(10pts) 5. Let $\{X_1,\ X_2,\ X_3,\ X_4\}$ be a random sample from the uniform distribution U(0,1) with the set of order statistics $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$.

(a) Calculate the distribution function of $X_{(3)}$, $F_{X_{(3)}}(x) = P(X_{(3)} \le x)$.

(b) Calculate the probability density function of $f_{X_{(3)}}(x)$.

(c) Calculate $E[X_{(3)}]$.

(Sa) $F(x) = F_{X_1}(x) = x, \quad 0 \le x \le 1$, then

$$\begin{aligned} F_{X_{(3)}}(x) &= P(X_{(3)} \le x) \\ &= \textstyle\sum_{k=3}^{4} \begin{pmatrix} a \\ b \end{pmatrix} [F(x)]^k [1 - F(x)]^{4-k} \\ &= 4x^3(1-x) + x^4 = 4x^3 - 3x^4, \quad 0 \le x \le 1 \end{aligned}$$

(Sb) $f_{X_{(3)}}(x) = 12x^2 - 12x^3, \quad 0 \le x \le 1$.

(Sc) $E[X_{(3)}] = \int_0^1 x f_{X_{(3)}}(x)dx = \int_0^1 12x^3 - 12x^4 dx = \frac{3}{5}$.

(10pts) 6. While rolling a balanced die of six faces successively, the first 3 occurred on the third roll. What is the expected number of rolls until the first 1 appears?

(Solution) Let $X,\ Y$ be the number of rolls until the first 1 appears and the first 3 occurs, respectively. We want to calculate

$$E(X|Y=3) = \sum_{x=1}^{\infty} x p_{X|Y}(x|y=3) = \sum_{x=1}^{\infty} x \frac{p(x,3)}{p_Y(3)},$$

where $p_Y(3) = (\frac{5}{6})^2(\frac{1}{6}) = \frac{25}{216}$, $p(1,3) = (\frac{1}{6})(\frac{5}{6})(\frac{1}{6}) = \frac{5}{216}$, $p(2,3) = (\frac{4}{6})(\frac{1}{6})(\frac{1}{6}) = \frac{4}{216}$, $p(3,3) = 0$, and for $x > 3$,

$$p(x,3) = (\frac{4}{6})^2(\frac{1}{6})(\frac{5}{6})^{x-4}(\frac{1}{6}) = \frac{1}{81}(\frac{5}{6})^{x-4}.$$

Therefore,

$$E(X|Y=3) = \sum_{x=1}^{\infty} x p_{X|Y}(x|y=3) = \sum_{x=1}^{\infty} x \frac{p(x,3)}{p_Y(3)} = \frac{157}{25} = 6\frac{7}{25}.$$