Exam 2 for EECS3030(02), Spring 2020

10:10-11:45 Weddnesday, June 17, 2020

Name : ______________ *SN* : ______________ *Index* : __________

(30pts) 1. Fill the following blanks.

(a) Let $\{X_i \sim b(16, 0.75),\ 1 \leq i \leq 10\}$ be a random sample from a binomial distribution. Define $X = \sum_{i=1}^{10} X_i$, then

the moment-generating function $M_X(t) =$ __________________, $Var(X) =$ _______

the probability mass function of X, $f_X(x) =$ ______________________________

(b) Let $\{Y_i,\ 1 \leq i \leq 8\}$ be a random sample of Poisson distribution with variance $Var(Y_8) = 2$ and define $Y = \sum_{i=1}^{8} Y_i$. Then

the moment-generating function $M_Y(t) =$ ________________________________

the p.m.f. of Y, $f_Y(y) =$ ____________________________, $Var(Y) =$ _________

(c) Let the joint probability distribution function of the lifetimes of two brands of mobile phones be given by $F(x, y) = (1 - e^{-x})(1 - e^{-y})\ \ if\ x > 0,\ y > 0$. Then the probability that one mobile phone lasts more than three times as long as the other is ____________

(d) Let the joint probability density function of random variables X and Y be given by $f(x, y) = 8xy, \quad 0 \leq x \leq y \leq 1$. Then

$f_X(x) =$ _______________, $E(Y) =$ ___________, $\rho(X, Y) =$ ______________

(e) Let the conditional probability density function of X, given that $Y = y$, be

$$f_{X|Y}(x|y) = \frac{x + y}{1 + y} e^{-x}, \quad 0 < x < \infty,\ 0 < y < \infty.$$

$P(X < 4|Y = 1) =$ _______________________

(30pts) 2. Fill the following blanks.

(a) Suppose that X and Y are independent and identically distributed exponential random variables with variance 4. Define $W = \frac{Y}{X+Y}$. Then

the probability density function $f_Y(y) =$ ____________________,

$E(W) =$ __________, $Var(W) =$ __________

(b) Let $\{X_i,\ 1 \le i \le 12\}$ be a random sample with the probability density function $f_{X_1}(x) = \frac{1}{16}x^2 e^{-x/2},\quad x > 0$. Define $W = \sum_{i=1}^{12} X_i$. Then

the moment-generating function $M_{X_3}(t) =$ __________, $Var(X_3) =$ __________

the moment-generating function $M_W(t) =$ ____________________

(c) Let $\{X_1, X_2, \cdots, X_9\}$ be a random sample of size *9* from the *exponential distribution* with $Var(X_4) = 4$. Define $Y = \sum_{i=1}^{9} X_i$. Then

the moment-generating function of Y, $\phi_Y(t) =$ ____________________,

the p.d.f. of Y, $f_Y(y) =$ ____________________, $Var(Y) =$ __________

(d) Let $Z \sim N(0,1)$. Define $Y = 3Z + 2$, then

the p.d.f. of Y, $f_Y(y) =$ ____________________

the moment-generating function of Y, $\phi_Y(t) =$ _______________, $Var(Y) =$ __________

(e) Let $\{X_i \sim b(12, 0.5),\ 1 \le i \le n\}$ be a random sample from a binomial distribution of size n. Define $W_n = (\sum_{i=1}^{n} X_i - 6n)/\sqrt{3n}$.

$E(X_n) =$ _______________, $Var(X_n) =$ _______________ According to the Central Limit Theorem,

the limiting moment-generating function, $limit_{n\to\infty} M_{W_n}(t) =$ ____________________

(10pts) 3. Let $\{X_i \sim N(3,4),\ 1 \le i \le n\}$ be a random sample of a normal distribution. Define $W = \sum_{i=1}^{n} \frac{X_i - 3}{2}$, $Y = \sum_{i=1}^{n} \left(\frac{X_i - 3}{2}\right)^2$.

(a) Write down the probability density function of X_n.

(b) Calculate the moment-generating function of W and its probability density function, $f_W(w)$, respectively.

(c) Calculate the moment-generating function of Y and its probability density function, $f_Y(y)$, respectively.

(d) Calculate $Var(W),\ Var(Y)$, respectively.

(10 pts) 4. The time it takes for a student to finish an aptitude test (in hours) has the probability density function

$$f(x) = 6(x-1)(2-x) \;\; if \; 1 < x < 2, \;\; and \;\; 0 \;\; elsewhere.$$

Approximate the probability that the average length of time it takes for a random sample of 16 students to complete the test in less than 1 hour and 45 minutes (that is, 1.75 hours).

(Hint) You can express your solution in terms of $\Phi(r)$, where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$, and $r \in (-4.99, 4.99)$.

(10pts) 5. Let $\{X_1,\ X_2,\ X_3,\ X_4\}$ be a random sample from the uniform distribution U(0,1) with the set of order statistics $X_{(1)} < X_{(2)} < X_{(3)} < X_{(4)}$.

(a) Calculate the distribution function of $X_{(3)}$, $F_{X_{(3)}}(x) = P(X_{(3)} \leq x)$.

(b) Calculate the probability density function of $f_{X_{(3)}}(x)$.

(c) Calculate $E[X_{(3)}]$.

(10pts) 6. While rolling a balanced die of six faces successively, the first 3 occurred on the third roll. What is the expected number of rolls until the first 1 appears?