Exam I for EECS3030(02), Spring 2020

10:10-11:45, May 6, 2020

Name : <u>*Instructor*</u> *SN* : <u>B14060</u> *Index* : <u>100</u>

(30pts) 1. Choose the *best* (unique) solution for each of the following problems.

(**2)(a)** A number is selected at random from the set $\{1, 2, \cdots, 30\}$. What is the probability that it is relatively prime to 180?

(1) $\frac{2}{15}$, (2) $\frac{4}{15}$, (3) $\frac{6}{15}$, (4) $\frac{8}{15}$, (5) none

(**5)(b)** The coefficient of the quadratic equation $ax^2 - 2bx + c = 0$ are determined by tossing a fair die three times (the first outcome is a, the second outcome is b, and the third one is c). The probability that the equation has two distinct real roots is

(1) $\frac{8}{216}$, (2) $\frac{25}{216}$, (3) $\frac{173}{216}$, (4) $\frac{193}{216}$, (5) none

(**4)(c)** How many divisors does 750 have?

(1) 8, (2) 10, (3) 12, (4) 16, (5) none

(**4)(d)** The number of nonnegative integer solutions (x_1, x_2, x_3) for $x_1 + x_2 + x_3 = 5$ is

(1) 6, (2) 10, (3) 15, (4) 21, (5) none

(**2)(e)** The coefficient of x^2y^3 in the binomial expansion of $(3x - 2y)^5$ is

(1) -1080, (2) -720, (3) 720, (4) 1080, (5) none

(**1)(f)** $\sum_{k=1}^{n} \binom{n}{k} 2^k =$

(1) $3^n - 1$, (2) $2^n - 1$, (3) 3^n, (4) 2^n, (5) none

(**1)(g)** $\sum_{k=1}^{n} \binom{n}{k} (-1)^k =$

(1) -1, (2) 0, (3) 1, (4) 2^n, (5) none

(**2)(h)** A list of all permutations of 13579 is put in an increasing order. The $100th$ number in the list is

(1) 91537, (2) 91573, (3) 93157, (4) 93175, (5) none

(5)**(i)** A number is selected at random from the set $\{1, 2, \cdots, 1000\}$. The probability that it is divisible neither by 3 nor by 5 is

(1) 0.60, (2) 0.50, (3) 0.40, (4) 0.30, (5) none

(2)**(j)** Let X be a randomly selected point from the interval $(0, 3)$. Then the probability that $P(X^2 - 3X + 2 > 0 \mid X)$ is

(1) $\frac{1}{3}$, (2) $\frac{2}{3}$, (3) $\frac{3}{4}$, (4) 1, (5) none

(10pts)2. Let the probability density function of X be $f(x) = \beta x^{\beta-1}, \quad 0 < x < 1, \quad 0 < \beta < \infty$. Show that $Y = -2\beta \ln(X)$ has an exponential distribution with $E(Y) = 2$.

Solution:

$$\begin{aligned} F(y) &= P(Y \leq y) &&= P(-2\beta ln(X) \leq y) \\ &= P(ln(X) \geq \tfrac{y}{-2\beta}) &&= P(X \geq e^{-y/2\beta}) \\ &= \textstyle\int_{e^{-y/2\beta}}^{1} \beta x^{\beta-1} dx &&= 1 - e^{-y/2} \end{aligned}$$

Then, $f(y) = F'(y) = \frac{1}{2}e^{-y/2}, \ y > 0$ which completes the proof.

(40pts)3. Fill the following blanks.

(a) Let A and B be independent events with $P(A) = 0.8$ and $P(B) = 0.6$, then
$P(A \cap B) = \underline{0.48}$, $P(A \cup B) = \underline{0.92}$

(b) Let a binomial random variable X have the probability mass function $f(x) = \binom{50}{x}(0.8)^x(0.2)^{50-x}$, $0 \le x \le 50$. Then
$E(X) = \underline{40}$, $Var(X) = \underline{8}$, $M(t) = \underline{(0.2 + 0.8e^t)^{50}}$

(c) If the moment-generating function of X is $M(t) = \frac{2}{5}e^t + \frac{1}{5}e^{2t} + \frac{2}{5}e^{3t}$, then
$E(X) = \underline{2}$, $Var(X) = \underline{\frac{4}{5}}$

(d) Let a random variable X have the Poisson distribution with variance 4. Then
$E(X) = \underline{4}$, $M(t) = \underline{Exp(4(e^t - 1))}$

(e) Let a random variable X have the geometric distribution with $E(X) = 2$. Then
$Var(X) = \underline{2}$, $M(t) = \underline{\frac{e^t}{2-e^t}}$

(f) A random variable X has p.d.f. $f(x) = \frac{1}{2}e^{-x/2}$, $x \ge 0$. Then
the 25th percentile of $X = \underline{2ln(4/3)}$, *the median of* $X = \underline{2ln(2)}$

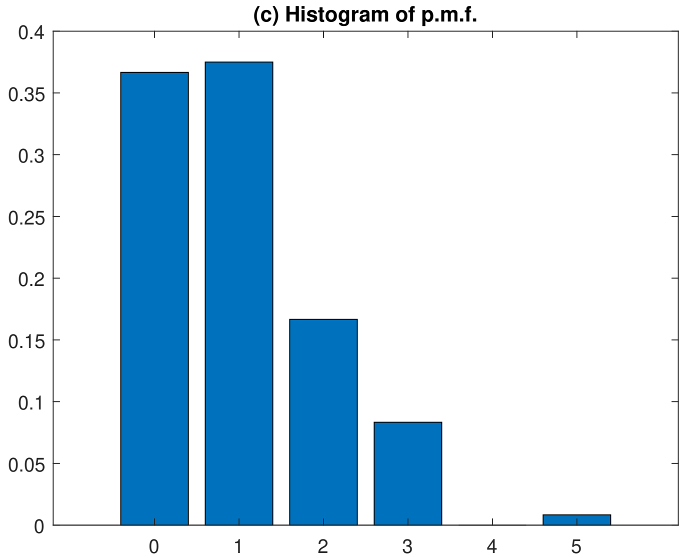
(g) Let $X \sim N(3, 1)$, then the p.d.f. of X is $f(x) = \underline{\frac{1}{\sqrt{2\pi}}Exp(-(x-3)^2/2)}$,

the moment-generating function $\phi_X(t) = \underline{Exp(3t + \frac{t^2}{2})}$

(h) Let a r.v. X have the probability density function $f(x) = \frac{1}{16}x^2e^{-x/2}$ $\forall\ x > 0$. Then
The moment-generating function $\phi_X(t) = \underline{\frac{1}{(1-2t)^3}}$, $Var(X) = \underline{12}$

(i) Let $Z \sim N(0, 1)$ be the standard normal distribution, define $Y = Z^2$. Then
The moment-generating function $\phi_Y(t) = \underline{\frac{1}{\sqrt{1-2t}}}$, $Var(Y) = \underline{2}$

(j) Define $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$, then $\Gamma(3) = \underline{2}$, $\Gamma(1.5) = \underline{\frac{\sqrt{\pi}}{2}}$



(10pts)4. Given a random permutation of the integers in the set $\{1, 2, 3, 4, 5\}$, let X be equal the number of integers that are in their natural position. Then the moment-generating function of X is

$$M(t) = \frac{44}{120} + \frac{45}{120}e^t + \frac{20}{120}e^{2t} + \frac{10}{120}e^{3t} + \frac{1}{120}e^{5t}$$

(a) Find $E(X)$ and $Var(X)$.

(b) Find the probability that at least one integer is in its natural position.

(c) Draw a graph of the histogram of the probability mass function of X.

Solution (a) $E(X) = M'(0) = 1$ and $Var(X) = M''(0) - (M'(0))^2 = 1$.

Solution (b) The probability that at least one integer is in its natural position is $1 - P(X = 0) = 1 - \frac{44}{120} = \frac{76}{120} = \frac{19}{30}$.

Solution (c) A graph of the histogram of the probability mass function of X is

(10pts)5. Suppose that 80% of the seniors, 70% of the juniors, 50% of the sophomores, and 30% of the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are seniors.

(a) What is the probability that a student uses the library frequently?

(b) If a student uses the library frequently, what is the probability that he or she is a sophomore?

(c) If a student uses the library frequently, what is the probability that he or she is a junior.

Solution (a) The probability that a student uses the library frequently is 0.55.

Solution (b) If a student uses the library frequently, the probability that he or she is a sophomore is $\frac{0.125}{0.55} = \frac{5}{22}$.

Solution (c) If a student uses the library frequently, the probability that he or she is a junior is $\frac{0.175}{0.55} = \frac{7}{22}$.