Solutions for Quiz 2, Spring 2020

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(5 pts)(1) Let R be the bounded region between $y = x$ and $y = x^2$. A random point (X, Y) is selected from R.

(a) Find the joint probability density function of X and Y, $f(x, y)$.

(b) Calculate the marginal probability density function $f_X(x)$.

(c) Calculate $E(X)$.

(Solution)

(Sa) $f(x, y) = c$, a constant, and $\int_0^1 \int_{x^2}^{x} f(x, y)dydx = 1$, so $f(x, y) = 6,\ \ 0 \le x, y \le 1.$

(Sb) $f_X(x) = \int_{x^2}^{x} 6dy = 6(x - x^2),\ 0 \le x \le 1.$

(Sc) $E(X) = \int_0^1 xf_X(x)dx = \int_0^1 (6x^2 - 6x^3)dx = \frac{1}{2}.$

(5 pts)(2) Let X and Y be independent exponential random variables both with mean 1. Let $W = max(X, Y)$.

(a) Find the distribution function $F_W(w)$ of W.

(b) Calculate $f_W(w)$.

(c) Calculate $E(W)$.

(Solution)

(Sa) $F_W(w) = P(W \le w) = P(X \le w,\ Y \le w) = P(X \le w) \times P(Y \le w) = (\int_0^w e^{-x}dx)(\int_0^w e^{-y}dy) = (1 - e^{-w})^2,\ w \ge 0.$

(Sa) $f_W(w) = F'_W(w) = 2e^{-w}(1 - e^{-w}),\ w \ge 0.$

(Sa) $E(W) = \int_0^\infty 2we^{-w}(1 - e^{-w})dw = 2 - \frac{1}{2} = \frac{3}{2}.$

(5 pts)(3) Let X and Y be independent and identically distributed exponential random variables with mean 3. Prove that $X/(X+Y)$ is *uniform* over $(0,1)$, that is, $X/(X+Y) \sim U(0,1)$.

(Solution)

(S3) Let $W = X/(X+Y)$, then $0 < W < 1$. For $0 < u < 1$, $F_(u) = P(W \leq u) = P(\frac{X}{X+Y} \leq u) = P(\frac{1-u}{u}X \leq Y) = \int_0^\infty \int_{\frac{1-u}{u}x}^\infty \frac{1}{3}e^{-x/3} \times \frac{1}{3}e^{-y/3} dydx = u$. Then, $f(u) = F'(u) = 1,\ 0 < u < 1$, hence $\frac{X}{X+Y} \sim U(0,1)$, a uniform distribution.

(5 pts)(4) Let X and Y be continuous random variables with the joint probability density function given by $f(x,y) = e^{-x(y+1)} \quad if \quad x > 0,\ 0 \leq y \leq e-1$, and $f(x,y) = 0,\ elsewhere.$

Calculate $E(X|Y=y)$.

(Solution)

(S4) Since $f_Y(y) = \int_0^\infty e^{-x(y+1)}dx = \frac{1}{1+y},\ 0 \leq y \leq e-1,\ f_{X|y}(x|y) = \frac{f(x,y)}{f_Y(y)} = (1+y)e^{-x(y+1)}$, then $E(X|Y=y) = \int_0^\infty x f_{X|y}(x|y)dx = \int_0^\infty x(1+y)e^{-x(y+1)}dx = \frac{1}{1+y}$.

(5 pts)(5) Let X and Y have the joint probability density function $f(x,y) = 1,\ if\ 0 \leq x,\ y \leq 1$.

(a) Calculate $P(X+Y \leq 3/4)$,

(b) Calculate $P(XY \leq 9/16)$,

(c) Calculate $P(Y \leq \sin(\pi X))$,

(Solution)

(Sa) $P(X+Y \leq 3/4) = \frac{9}{32}$,

(Sb) $P(XY \leq 9/16) = \frac{3}{4} \times \frac{3}{4} + 2\int_{\frac{3}{4}}^1 \frac{9}{16x}dx = \frac{9}{16} + \frac{9}{8}ln(\frac{4}{3})$,

(Sc) Calculate $P(Y \leq \sin(\pi X)) = \int_0^1 sin(\pi x)dx = \frac{2}{\pi}$,