Quiz 1, Spring 2020

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$Name:$ $\underline{Chaur-Chin\ Chen}$ $SN:$ $\underline{5731078}$ $Index:$ $\underline{00}$

(10pts) 1. Solve the following problems by showing the brief procedures.

(a) A number is selected at random from the set $\{1, 2, \cdots, 63\}$. What is the probability that it is relatively prime to 63?

(b) A number is selected at random from the set $\{1, 2, \cdots, 1000\}$. What is the probability that it is neither divisible by 3 nor by 5?

(c) The coefficients of the quadratic equation $x^2 + bx + c = 0$ are determined by tossing a fair die twice (the first outcome is b, the second one is c). Find the probability that the equation has real roots.

(d) Find the number of *positive* integer solutions (x_1, x_2, x_3) for $x_1 + x_2 + x_3 = 8$.

(e) Calculate the coefficient of x^2y^3 in the binomial expansion of $(2x - 3y)^5$.

Ans(a) $\frac{\phi(63)}{63} = \frac{63\times(1-1/3)(1-1/7)}{63} = \frac{36}{63} = \frac{4}{7}$.

Ans(b) $P(D_3' \cap D_5') = 1 - P(D_3 \cup D_5) = 1 - \frac{333}{1000} - \frac{200}{1000} + \frac{66}{1000} = \frac{533}{1000}$.

Ans(c) $S = \{(b, c) \mid 1 \le b, c \le 6\}$, $E = \{(b, c) \in S \mid b^2 - 4c \ge 0\}$, $P(E) = \frac{|E|}{|S|} = \frac{19}{36}$.

Ans(d) $C(8-1, 3-1) = C(7, 2) = 21$.

Ans(e) $C(5, 2)(2)^2(-3)^3 = -1080$.

(3pts) 2. For any positive integer n, prove that $\begin{pmatrix} 2n \\ n \end{pmatrix} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^2$.

Ans 2. Consider the coefficient of x^n in $(1+x)^{2n} = (1+x)^(1+x)^n$, the LHS has the coefficeint $\begin{pmatrix} 2n \\ n \end{pmatrix}$, the RHS has the coefficient $\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n \\ n-k \end{pmatrix} = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^2$

(3pts) 3. Evaluate $\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} + \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}$.

Ans 3. $\sum_{i=0}^{n} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} + \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} = (1-1)^n + (1+1)^2 = 2^n$.

(3pts) 4. Suppose that 100 freshmen CS students took both Calculus and Physics in the same class, 70% passed Calculus, 60% passed Physics, and 50% passed both. If a randomly selected freshmen is found to have passed Calculus, what is the probability that he or she also passed Physics?

Ans 4. Let C be the event that a student passed Calculus, and let S be that a student passed Physics, then $P(C) = 0.7,\ P(S) = 0.6,\ and\ P(S \cap C) = 0.5.\ P(S \mid C) = 0.5/0.7 = 5/7.$

(3pts) 5. In a certain factory, machines A, B, C are producing masks of the same style and size. Of their production, machines A, B, C produce 2%, 1%, and 3% defective masks, respectively. Of the total production of masks, machine A produces 35%, machine B produces 25%, and machine C produces 40% masks, respectively. If one mask is randomly selected and is defective, calculate the probability that it is produced from machine A, machine B, and machine C, respectively.

Ans 5. Let D be the event that a mask is defective, and A, B, C be the event that of machine A, B, C production, respectively. Then $P(D|A) = 2\%$, $P(D|B) = 1\%,\ and\ P(D|C) = 3\%$, and $P(A) = 35\%$, $P(B) = 25\%$, $P(C) = 40\%$, and. $P(D) = P(D|A)P(A)+P(D|B)P(B)+P(D|C)P(C) = 0.02*0.35+0.01*0.25+0.03*0.40 = 0.0215$. Thus, $P(A|D) = \frac{70}{215} = \frac{14}{43}$, $P(B|D) = \frac{25}{215} = \frac{5}{43}$, $P(C|D) = \frac{120}{215} = \frac{24}{43}$.

(3pts) 6. Show your procedures to compute the following integrals.

(a) $\gamma = \int_0^\infty e^{-x^2/2}dx$.

(b) $\Gamma(\frac{1}{2})$, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$

Ans (a)

$$\begin{aligned}\gamma^2 &= \int_0^\infty e^{-x^2/2}dx \int_0^\infty e^{-y^2/2}dy &&= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)/2}dxdy \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2/2}rdrd\theta &&= \frac{\pi}{2}\end{aligned}$$

Then $\gamma = \frac{\sqrt{\pi}}{\sqrt{2}}$.

Ans (b) $\Gamma(\frac{1}{2}) = \int_0^\infty e^{-t}t^{-\frac{1}{2}}dt = \sqrt{\pi}$.