Chapters 8.1-3,9.1. Bivariate and Multivariate Distributions

- 8.1 Joint Distributions of Two Random Variables
- 8.2 Independent Random Variables
- 8.3 Conditional Distributions

Definition 8.1 on P.329:

Let X and Y be two discrete random variables defined on the same sample space. Let the sets of possible values of X and Y be A and B, respectively. The function

$$p(x, y) = P(X = x, Y = y)$$

is called the *joint probability mass function of X and Y. Note that* $p(x, y) \geq 0$.

If $x \notin \mathrm{A}$ or $\mathrm{y} \notin B$, *then* $p(x, y) = 0$. Also $\sum_{x \in A} \sum_{y \in B} p(x, y) = 1$.

Marginal Probability Mass Functions p_X, p_Y

Definition 8.2 on P.329-330:

Let X and Y have *joint probability mass function* $p(x, y)$. Let A be the set of possible values of X and B be the possible values of Y. Then the functions $p_X(x) = \sum_{y \in B} p(x, y)$ and $p_Y(y) = \sum_{x \in A} p(x, y)$ are called the marginal probability mass functions of X and Y, respectively.

(8.1) on P.330

A small college has 90 male and 30 female professors. An ad hoc committee of five is selected at random to write the vision and mission of the college. Let X and Y be the number of men and women on this committee, respectively. Then

(a) The joint probability mass function of X and Y is

$$p(x, y) = \frac{\binom{90}{x}\binom{30}{y}}{\binom{120}{5}}, \text{ if } x, y \in \{0, 1, 2, 3, 4, 5\} \ \ and \ \ x + y = 5, \quad p(x, y) = 0 \ \ otherwise.$$

(b) $p_X(x) = \frac{\binom{90}{x}\binom{30}{5-x}}{\binom{120}{5}}, \quad p_Y(y) = \frac{\binom{90}{5-y}\binom{30}{y}}{\binom{120}{5}}, \quad x, y \in \{0, 1, 2, 3, 4, 5\}$

Example 8.2 on P.330-331

(8.2) Roll a balanced die and let the outcome be X. Then toss a fair coin X times and let Y denote the number of tails. What is the joint probability mass function $p(x, y)$ of X and Y and the marginal probability mass functions of X and Y?

Ans: Clearly, $X \in \{1,2,3,4,5,6\}, Y \in \{0,1,2,3,4,5,6\}$.

List a table for $p(x, y), where\ p(1,0) = \frac{1}{12}, p(1,1) = \frac{1}{12},$

$p(1, y) = 0\ if\ y \geq 2;\ \ and\ p_X(x) = \frac{1}{6}\ for\ x \in \{1,2,3,4,5,6\}.$

We can also compute E(X) and E(Y) by

$$E(X) = \sum_{x \in A} x p_X(x), \text{ and } E(Y) = \sum_{y \in B} y p_Y(y).$$

Example 8.3 on P.331-332

(8.3) Let the joint probability mass function of random variables X and Y be given by

$$p(x,y) = \begin{cases} \frac{1}{70}x(x+y) & if\ x = 1,2,3,\ \ y = 3,4 \\ 0 & elsewhere \end{cases}$$

Then $p_X(x) = \frac{1}{35}x^2 + \frac{1}{10}x,\ \ x = 1,2,3.$ E(X)=$\frac{17}{7} \approx 2.43.$

$p_Y(y) = \frac{1}{5} + \frac{3}{35}y,\ \ y = 3,4.\ \ E(Y) = \frac{124}{35} \approx 3.54.$

Theorem 8.1 on P.332

(Theorem 8.1) Let $p(x,y)$ be the joint probability mass function of discrete random variables $X\,and\,Y$. If $h: R^2 \rightarrow$R is a function of two variables. Then h(X,Y) is a discrete random variable with the expected value given by

$$E[h(X,Y)] = \sum_{x \in A} \sum_{y \in B} h(x,y)p(x,y)$$

provided the sum is absolutely convergent.

Corollary

Let h(x,y)=x+y, then E(X+Y)=E(X)+E(Y).

Example 8.4 on P.333

(8.4) Let the joint probability mass function of the discrete random variables X and Y be given by

$$p(x, y) = \frac{2}{11}\left(\frac{x}{y}\right) \;\; if\ x = 1{,}2,\ \ y = 1{,}2{,}3; \quad and\ 0\ elsewhere.$$

Then

$$E(XY) = \sum_{x=1}^{2}\sum_{y=1}^{3} xyp(x, y) = \sum_{x=1}^{2}\sum_{y=1}^{3} xy \times \frac{2}{11}\frac{x}{y} =$$

$$\sum_{x=1}^{2}\sum_{y=1}^{3}\frac{2}{11}x^{2} = \frac{2}{11}\sum_{x=1}^{2} x^{2}\sum_{y=1}^{3} 1 = \frac{2}{11}\sum_{x=1}^{2} 3x^{2} = \frac{30}{11}$$

Joint Probability Density Functions on P.333

Definition 8.3 on P.333

Two random variables X and Y, defined on the same sample space, have a continuous joint distribution if there exists a nonnegative function of two variables, $f(x, y)$, on $R \times R$, such that for any region in the $xy-plane$ that can be formed from rectangles by a countable number of set operations,

$$P\big((X, Y) \in R\big) = \iint_{R} f(x, y)dxdy \qquad (8.2)$$

The function $f(x, y)$ is called the joint probability *density* function of X and Y.

Let $R = \{(x, y)\colon\ x \in A, y \in B\}, then,$ eq.(8.2) gives

$$\int_{y \in B} \int_{x \in A} f(x, y)dxdy \qquad (8.3)$$

(a) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$

(b) $P(X = a, Y = b) = \int_{a}^{a} \int_{b}^{b} f(x, y)dydx = 0$

(c) $P(a \le X < b,\ c < Y < d) = \int_{c}^{d} \int_{a}^{b} f(x, y)dxdy$

(d) $P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$.

Marginal Probability Density Functions

- $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx, \;\; f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$

Definition 8.4 on P.335

Let X and Y have joint probability density function $f(x,y)$; then the $functions$ f_X and f_Y defined above are called marginal probability density functions of X and Y, respectively.

The joint distribution function of X and Y is defined as

$F(u,v) = P(X \le u, Y \le v)$ for all $-\infty < u, v < \infty$. Then

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

Example 8.5 onP.336

(8.5)

$f(x,y) = \gamma xy^2 \quad 0 \le x \le y \le 1, \quad 0 \;\; otherwise.$

(a) Determine γ. Since $\int_0^1 \int_x^1 \gamma xy^2 \, dydx = \frac{\gamma}{10}$=1, then $\gamma = 10$.

(b) $f_X(x) = \int_x^1 10xy^2 \, dy = \frac{10}{3} x(1 - x^3), \; 0 \le x \le 1.$

(c) $f_Y(y) = \int_0^y 10xy^2 \, dx = 5y^4, \; 0 \le y \le 1.$

(d) Calculate E(X)=$\frac{5}{9}$ and E(Y)=$\frac{5}{6}$.

Example 8.6 on P.337

For $\lambda > 0, let$

$F(x, y) = 1 - \lambda e^{-\lambda(x+y)}$ if $x > 0,\ y > 0,$

$F(x, y) = 0 \quad otherwise.$

Determine if F is the joint distribution function of

two random variables X and Y. (No, because $\frac{\partial^2}{\partial x \partial y} F(x, y)$ <0)

Example 8.7 and Definition 8.5 on P.338-9

(8.7) A circle of radius 1 is inscribed in a square with sides length 2. A point selected at random from the square. What is the probability that it is inside the circle? Note that by a point being elected at random from the square we mean that the point is selected that all subsets of equal areas of the square are equally likely to contain the point.

Ans: $\frac{\pi}{4}$

Definition 8.5 *Let S be the subset of the plane with area A(S). A point is said to be randomly selected from S if for any subset R of S with area A(R), the probability that R contains the point is A(R)/A(S).*

Example 8.8 on P.339-340

[7.3, P.282-283]

What is the probability that a random chord of a circle is larger than a side of an equilateral triangle inscribed into the circle? Ans:(1/2, 1/3, ¼)

(8.8) A man invites his fiancée to a fine hotel for a Sunday brunch.. They decide to meet in the lobby of the hotel between 11:30 A.M. and 12:00 noon. If they arrive at random time during this period, what is the probability that they will meet within 10 minutes?(等待小於十分鐘)

Ans: Let S=$\{(x, y): 0 \leq x, y \leq 30\}, R = \{(x, y): |x - y| \leq 10\}$.

P(R|S)=$\frac{5}{9}$

Theorem 8.2 and Example 8.10 on P.341-342

Theorem 8.2 Let $f(x,y)$ be the joint probability density function of random variables X and Y. If h is a function of two variables from R^2 to R, then $h(X,Y)$ is a random variable with the expected value given by

$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)f(x,y)dxdy,$$

if the integral is absolutely convergent.

Corollary For random variables X and Y, E(X+Y)=E(X)+E(Y).

(8.10) Let $f(x,y) = \frac{3}{2}(x^2 + y^2)\ \ if\ 0 < x,\ \ y < 1$. Then

$$E(X^2 + Y^2) = \frac{14}{15}.$$

Questions on P.342-349: A1,3,11,13,17;Q1,Q2

(A1) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = k\left(\frac{x}{y}\right) if\ x = 1,2;\ \ y = 1,2.$$

Then

(a) The constant $k = 2/9$.

(b) The $marginal\ p.m.f.$ $p_X(x) = \frac{x}{3}, x = 1,2;\ \ p_Y(y) = \frac{2}{3y}, y = 1,2.$

(c) P(X>1|Y=1)=$\frac{P(X>2,Y=1)}{P_Y(y=1)} = \frac{p(2,1)}{p_Y(1)} = \frac{(\frac{2}{9})(\frac{2}{1})}{2/(3\times 1)}$=$\frac{2}{3}$.

(d) E(X)=5/3, and E(Y)=4/3.

Exercise A3 on P.343

(A3) Let the joint probability mass function of discrete random variables X and Y be given by

$$p(x,y) = k(x^2 + y^2) \;\; if \; (x,y) = (1,1),(1,3),(2,3)$$

(a) Then the constant $k = 1/25$.

(b) The $marginal\ probability\ mass\ functions\ \ p_X(x), p_Y(y)\ are$

$$p_X(x) = \frac{12}{25} if\ x = 1, \frac{13}{25}\ if\ x = 2.$$

$$p_Y(y) = \frac{2}{25} if\ y = 1, \frac{23}{25}\ if\ y = 3.$$

(c) E(X)=25/38, and E(Y)=71/25.

Exercise A11 on P.344

(A11) Let the joint probability density function of random variables X and Y be given by

$$f(x,y) = 8xy \;\; if \; 0 \le y \le x \le 1.$$

Show that

(a) $f_X(x) = \int_0^x 8xydy = 4x^3, 0 \le x \le 1,$

$f_Y(y) = \int_y^1 8xydx = 4y - 4y^3, 0 \le y \le 1.$

(b) Compute E(X)=4/5, and E(Y)=8/15.

Exercise A13 on P.344

(A13) Let X and Y have the joint probability density function

$$f(x, y) = 1 \;\; if \; 0 \le x \le 1, 0 \le y \le 1.$$

Show that

(a) $P(X+Y \le \frac{1}{2}) = \frac{1}{8}$,

(b) $P\left(X - Y \le \frac{1}{2}\right) = \frac{7}{8}$,

(c) $P\left(XY \le \frac{1}{4}\right) = \frac{1}{2}\ln(2) + \frac{1}{4}$,

(d) $P(X^2 + Y^2 \le 1) = \frac{\pi}{4}$

8.2 Independent Random Variables on P.348-360

Definition: Two random variables are called *independent*, if for arbitrary subsets *A and B of* R,

$$P(X \in A \subseteq R, Y \in B \subseteq R) = P(X \in A)P(Y \in B)$$

Theorem 8.3 on P.349

Let X and Y be two random variables defined on the same sample space. If F is the joint distribution function of X and Y, then X and Y are independent iff for all real numbers t and u,

$$F(t,u) = F_X(t)F_Y(u)$$

Theorem 8.4 Let X and Y be two discrete random variables defined on the same sample space. X and Y are independent *if and only (iff)*

$$p(x,y) = p_X(x)p_Y(y).$$

Example 8.11 on P.349

(8.11) Suppose that 4% of the bicycle fenders, produced by a stamping machine from the strips of steel, need smoothing. What is the probability that, of the next 13 bicycle fenders stamped by this machine, two need smoothing and, of the next 20, three need smoothing?

Ans: Let X be the number of bicycle fenders among the first 13 that need smoothing. Let Y be the number of those among the next 7 that need smoothing. We want to compute P(X=2, Y=1)=P(X=2)P(Y=1), where X and Y are independent binomial r.v.s X~b(13,0.04), Y~b(7,0.04).

So P(X=2,Y=1)= $\binom{13}{2}(0.04)^2(0.96)^{11}\binom{7}{1}(0.04)^1(0.96)^6$

Theorem 8.5 on P.350

(Theorem 8.5) Let X and Y be independent random variables. Then for the real-valued functions $g: R \to R$ and $h: R \to R$, $g(X) and\ h(Y)$ are also independent random variables.

Proof: It suffices to show that for any two real-valued numbers a, b,

$$P(g(X) \leq a, h(Y) \leq b) = P(g(X) \leq a)P(h(Y) \leq b).$$

Let $A = \{x: g(x) \leq a\}$ and $B = \{y: h(y) \leq b\}$. Clearly, $x \in A\ \ iff\ \ g(x) \leq a$ and $y \in B\ \ iff\ \ h(y) \leq b$. Therefore,

$P(g(X) \leq a, h(Y) \leq b) = P(X \in A, Y \in B) = P(X \in A)P(Y \in B) = P(g(X) \leq a)P(h(Y) \leq b)$.

By this theorem, if X and Y are independent, then the sets of $\{X^2,\ Y\}$,

{sinX, e^Y} are also independent.

Theorem 8.6 and Example 8.12 on P.350

(Theorem 8.6) Let X and Y be independent random variables. Then for all real-valued function $g: R \to R$ and $h: R \to R$,

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)],$$

where we assume that $E[g(X)]$ and $E[h(Y)]$ are finite.

Proof: $E[g(X)h(Y)] = \sum_{x \in A} \sum_{y \in B} g(x)h(y)p(x,y) = \sum_{x \in A} \sum_{y \in B} g(x)h(y)p_X(x)p_Y(y) = \cdots$

Note that if X and Y are independent, then E(XY)=E(X)E(Y), however, the converse is not always true.

(Example 8.12) Let X be a random variable with the set of possible values {-1, 0, 1} and probability mass function p(-1)=p(0)=p(1)=1/3. Letting $Y = X^2$, then E(X)=0, E(Y)=2/3, and E(XY)=0=E(X)E(Y). Whereas X and Y are dependent.

Independence of Continuous Random Variables

(Theorem 8.7) Let X and Y be jointly continuous random variables with joint probability density function $f(x, y)$. Then X and Y are independent iff $f(x, y) = f_X(x) f_Y(y)$.

(8.13) A point is selected at random from the rectangle

$$R = \{(x, y): 0 < x < a, 0 < y < b\}.$$

Let X be the *x*-coordinate and Y be the *y*-coordinate of the point selected. Show that X and Y are independent r.v.s.

Proof: $f(x, y) = \frac{1}{ab},\ if\ (x, y) \in R = \{(x, y): 0 < x < a, 0 < y < b\}$.

$$f_X(x) = \frac{1}{a},\ x \in (0, a),\ and\ f_Y(y) = \frac{1}{b}, y \in (0, b).$$ Hence,

$$f(x, y) = f_X(x) f_Y(y)\ for\ (x, y) \in R$$

Examples 8.14, 8.16 on P.352-355.

(8.14) Stores A and B, which belong to the same owner, are located in two different towns. If the probability density function of the weekly profit of each store, in thousand dollars, is given by

$$f(x) = \frac{x}{4} \; if \; 1 < x < 3, \qquad 0 \; otherwise.$$

and the profit of one store is independent of the other, what is the probability that next week one store makes at least \$500 more than the other store?

Ans: 2P(X>Y+500/1000)=549/1024, where f(x,y)=(x/4)(y/4).

(8.16) Prove that two random variables X and Y with joint probability density function $f(x,y) = 8xy,\ 0 \le x \le y \le 1$ *are not independent*. Because $f(x,y) \neq f_X(x) f_Y(y), where$

$$f_X(x) = \int_x^1 8xydy = 4x - 4x^3, \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_0^y 8xy\,dx = 4y^3, \qquad 0 \le y \le 1$$

Properties of Independent Random Variables

- $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ if $X\ and\ Y\ are\ independent$.
- If X and Y are independent, E(XY)=E(X)E(Y), but the converse is not necessarily true.

Exercises A1,A3,7,15,19; B27,29 from P.356-360

(A1) Let the joint probability mass function of random variables X and Y be given by

$$p(x,y) = \frac{1}{25}(x^2 + y^2)\ if\ x = 1,2, y = 0,1,2$$

Then, X and Y are not independent because $p(x,y) \neq p_X(x)p_Y(y)$.

(A3) Let X and Y be independent r.v.s each having the probability mass function

$$p(x) = \frac{1}{2}\left(\frac{2}{3}\right)^x,\ x = 1,2,3,\cdots$$

Then P(X=1, Y=3)=P(X=1)P(Y=3)= $\frac{1}{2}\left(\frac{2}{3}\right)^1 \frac{1}{2}\left(\frac{2}{3}\right)^3$ = $\frac{4}{81}$, and

P(X+Y=3)=P(X=1,Y=2)+P(X=2,Y=1)= $\frac{4}{27}$.

Exercises A7, A17, A19 from P.356-360

(A7) Let X and Y be two independent r.v.s with distribution functions F and G, respectively. Then, the distribution functions of max(X,Y) and min(X,Y) are

$P(\max(X,Y) \le t) = P(X \le t, Y \le t) = P(X \le t)P(Y \le t) = F(t)G(t),$

$P(\min(X,Y) \le t) = 1 - P(\min(X,Y) > t) = 1 - \left(1 - P(X \le t)\right)\left(1 - P(Y \le t)\right)$

$= F(t) + G(t) - F(t)G(t)$

(A17) Let X and Y be independent points *randomly selected from the interval (-1,1).* Then $E[\max(X,Y)] = \frac{1}{3}$.

(A19) A point is selected at random from the disk

$$R = \{(x,y) \in R^2 \colon x^2 + y^2 \le 1\}.$$

Let X be the $x - coordinate$ and Y be the $y - coordinate$ of the point selected. Determine if X and Y are independent random variables.

(Ans) No, it is not independent because $f(x,y) \ne f_X(x)f_Y(y)$.

Exercises B27, B29, A15 from P.356-360

(B27) Let the joint probability density function of two random variables X and Y satisfy $f(x, y) = g(x)h(y),\ -\infty < x, y < \infty$, where $g\ and\ h\ are\ two\ functions\ from\ R\ to\ R$. Show that X and Y are independent. (Hint:) Show that $f(x, y) = f_X(x)f_Y(y)$.

(B29) Suppose that X and Y are independent and identically distributed exponential random variables with mean $1/\lambda$. Prove that X/(X+Y) is uniform over (0,1). (Hint:) Show that P((X/X+Y) $\leq u) = u,\ u \in (0,1)$.

(A15) Let X and Y be two independent r.v.s with the same p.d.f. $f(x) = e^{-x}, x > 0$. Show that g, the p.d.f. of X/Y, is given by $g(t) = 1/(1+t)^2$ if $0 < t < \infty$. (Hint:) G(t)=P((X/Y)<t), and g(t)=G'(t).

8.3 Conditional Distributions

- $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)}$ if $p_Y(y) > 0$
- $p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}$ if $p_X(x) > 0$

$$F_{X|Y}(x|y) = P(X \leq x|Y = y)$$
$$= \sum_{t \leq x} P(X = t|Y = y) = \sum_{t \leq x} p_{X|Y}(t|y)$$

$$E(h(X)|Y = y) = \sum_{x \in A} h(x) p_{X|Y}(x|y)$$

Example 8.17 on P.361

Let the joint probability mass function of X and Y are given by

$$p(x,y) = \frac{1}{15}(x+y), \qquad if\ x = 0, 1, 2, y = 1,2;\ \ 0\ otherwise.$$

Then

(a) $p_Y(y) = \sum_{x=0}^{2} p(x,y) = \frac{1+y}{5}$, y = 1,2.

(b) $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} = \frac{(x+y)/15}{(1+y)/5} = \frac{x+y}{3(1+y)},\ \ x = 0,1,2,;\ y = 1\ or\ 2.$

(c) P(X=0|Y=2)=$\frac{0+2}{3(1+2)} = \frac{2}{9}$.

Example 8.18 on P.362

(8.18) Let *N(t)* be the number of males who enter a certain post office at or prior to time t and let *M(t)* be the number of females who enter a certain post office at or prior to time t. Suppose that $\{N(t): t \geq 0\}$ and $\{M(t): t \geq$

Example 8.19 on P.363

(8.19) Calculate the expected number of aces in a randomly selected poker hand that is found to have exactly two jacks.

Ans: Let X and Y be the number of aces and jacks in a random poker hand, respectively. Then

E(X|Y=2)=$\sum_{x=0}^{3} x p_{X|Y}(x|2) = \sum_{x=0}^{3} x \frac{\binom{4}{x}\binom{44}{3-x}}{\binom{48}{3}} \approx 0.25$

Note that: $E(h(X)|Y = y) = \sum_{x \in A} h(x) p_{X|Y}(x|y)$

Example 8.20 on P.364

(8.20) While rolling a balanced die successively, the first 6 occurred on the third roll. What is the expected number of rolls until the first 1?

Ans: Let X and Y be the number of rolls until the first 1 and first 6, respectively. The required quantity to compute is

$$E(X|Y=3)=\sum_{x=1}^{\infty} x p_{X|Y}(x|3)=\sum_{x=1}^{\infty} x \frac{p(x,3)}{p_Y(3)}=$$

$$\frac{1}{\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)}\sum_{x=1}^{\infty} x p(x,3)=\cdots=6.28$$

Conditional Continuous Distributions on P.366

$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$, $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$

$$F_{X|Y}(x|y) = P(X \le x|Y = y) = \int_{-\infty}^{x} f_{X|Y}(t|y)dt$$

(8.22) Let X and Y be continuous random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2 + y^2) \; if \; 0 < x < 1, 0 < y < 1 \\ 0, \qquad otherswise \end{cases}$$

$$f_Y(y) = \int_0^1 \frac{3}{2}(x^2 + y^2)\, dx = \frac{1}{2}(1 + 3y^2), \qquad 0 < y < 1.$$

Then $f_{X|Y}(x|y) = \frac{3(x^2+y^2)}{1+3y^2}$ for $0 < x < 1, 0 < y < 1.$

Examples 8.23-8.26 on P.367-370.

(8.23) First, a point Y is selected at random from the interval (0,1). Then another point X is chosen at random from (0,Y). Find the probability density function of X, $f_X(x)$.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) dy$$

$$= \int_x^1 (\frac{1}{y} \times 1) dy = -\ln(x)\,.$$

Since $f_Y(y) = 1\ if\ 0 < y < 1$, and

$$f_{X|Y}(x|y) = \frac{1}{y}\ \ if\ 0 < y < 1, 0 < x < y$$

Example 8.24 on P.368

(8.24) Let the conditional probability density function of X, given $Y = y$, be

$$f_{X|Y}(x|y) = \frac{x+y}{1+y}e^{-x}, \qquad 0 < x < \infty, 0 < y < \infty.$$

Then $P(X < 1|Y = 2) = \int_0^1 \frac{x+2}{1+2}e^{-x}dx = 1 - \frac{4}{3}e^{-1} \approx 0.509$

Note that: $E(X|Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx,$

$$E(h(X)|Y = y) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y)dx$$

Example 8.25 on P.369

(8.25) Let X and Y b continuous random variables with joint probability density function

$f(x,y) = e^{-y}\ if\ y > 0,\ 0 < x < 1,$ 0 elsewhere.

Then

$$E(X|Y=2) = \int_{-\infty}^{\infty} x f_{X|Y}(x|2)dx = \int_0^1 x \frac{f(x,2)}{f_Y(2)} dx = \int_0^1 x \frac{e^{-2}}{e^{-2}} dx = \frac{1}{2},$$

since $f_Y(2) = \int_0^1 f(x,2)dx = \int_0^1 e^{-2} dx = e^{-2}$

Example 8.26 on P.369-370

(8.26) The lifetimes of batteries manufactured by a certain company are identically distributed with distribution and probability density functions *F and f,* respectively. In terms *of F, f, and s*, find the expected value of the lifetime of an s-hour-old battery.

Let X be the lifetime of the s-hour-old battery. We want to calculate $E(X|X > s)$. Note that $F_{X|X>s}(t) = P(X \le t|X > s)$, and

$f_{X|X>s}(t) = F'_{X|X>s}(t)$.

Then $E(X|X > s) = \cdots = \frac{1}{1-F(s)} \int_s^\infty tf(t)dt$.

Remark: $E(X|X < s) = \cdots = \frac{1}{F(s)} \int_0^s tf(t)dt$.

8.4 Transformation of two random variables

(Theorem 8.8) on P.374

Let X and Y be continuous random variables with joint probability density function $f(x,y)$. Let h_1, h_2 be real-valued functions of two variables, $U = h_1(X,Y), V = h_2(X,Y)$. Suppose that

$\begin{cases} h_1(x,y) = u \\ h_2(x,y) = v \end{cases}$ has a unique solution for R=$\{(x,y)\} \to Q = \{(u,v)\}$, say,

$\begin{cases} x = w_1(u,v) \\ y = w_2(u,v) \end{cases}$ and Jacobian J=$\begin{vmatrix} \frac{\partial w_1}{\partial u} & \frac{\partial w_1}{\partial v} \\ \frac{\partial w_2}{\partial u} & \frac{\partial w_2}{\partial v} \end{vmatrix} \neq 0$, then U, V have the joint probability density function

$$g(u,v) = f\big(w_1(u,v), w_2(u,v)\big)|Jacobian|,\ (u,v) \in Q.$$

Example 8.28 on P.376

(8.28) Let X and Y be two independent uniform random variables over (0,1); show that the random variables

$U = \cos(2\pi X)\sqrt{-2lnY}$ and $V = \sin(2\pi X)\sqrt{-2lnY}$ are two independent standard normal random variables.

Proof: $Y = \exp[-\frac{1}{2}(U^2 + V^2)], X = \frac{1}{2\pi} tan^{-1}\left(\frac{V}{U}\right)$, then the Jacobian

$$J_{u,v}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{2\pi}\exp(-(u^2+v^2)/2) \neq 0.$$

$$g(u,v) = f(x,y)|Jacobian| = \frac{1}{2\pi}\exp(-(u^2+v^2)/2)$$

Theorem 8.9 Convolution Theorem on P.378

Let X and Y be continuous independent r.v.s with probability density functions f_1, f_2 and distribution functions F_1 and F_2, respectively. Then $g\ and\ G$, the probability density and distribution functions of X+Y, respectively, are given by

$$g(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x)dx,$$

$$G(t) = \int_{-\infty}^{\infty} f_1(x) F_2(t-x)dx$$

Let U=X+Y, V=X. Then the *Jacobian* $J_{u,v}(x,y) = -1 \neq 0.$

9.1/2. Multivariate Distributions (P.393-417)

(Definition 9.1) on P.393

Let $X_1, X_2, \cdots, X_n$ be discrete random variables defined on the same sample space, with sets of possible values $A_1, A_2, \cdots, A_n$, respectively. The function

$$p(x_1, x_2, \cdots, x_n) = P(X_1, X_2, \cdots, X_n)$$

is called the joint probability mass function of $X_1, X_2, \cdots, X_n$. Note that

$(a)\ p(x_1, x_2, \cdots, x_n) \geq 0$

$(b)\ If\ for\ some\ i, 1 \leq i \leq n, x_i \notin A_i, then\ p(x_1, x_2, \cdots, x_n)$=0

$(c)\ \sum_{x_j \in A_j, j \neq i} p(x_1, x_2, \cdots, x_n) = 1$

Marginal Probability Mass Function $p_{X_i}(x_i)$

- $p_{X_i}(x_i) = P(X_i = x_i) = \sum_{x_j \in A_j, j \neq i} p(x_1, x_2, \cdots, x_n)$ (Eq. 9.1)

(9.1) Dr. Shams has 23 hypertensive patients, of whom five do not use any medicine but try to lower their blood pressures by self-help: dieting, exercise, not smoking, relaxation, and so on. Of the remaining 18 patients, 10 use beta blockers and 8 use diuretics. A random sample of seven of all these patients is selected. Let X, Y, Z be the number of the patients in the sample trying to lower their blood pressures by self-help, beta blockers, and diuretics, respectively. Find the joint probability mass function and marginal probability mass functions of X, Y, Z.

$$p(x,y,z) = \frac{\binom{5}{x}\binom{10}{y}\binom{8}{z}}{\binom{23}{7}},\ 0 \leq x \leq 5, 0 \leq y \leq 7,\ 0 \leq z \leq 7,\ x + y + z = 7.$$

Joint and Marginal Distribution Functions

(9.2) $F(t_1, t_2, \cdots, t_n) = P(X_1 \leq t_1, X_2 \leq t_2, \cdots, X_n \leq t_n)$

(9.3) $F_{X_i}(t_i) = P(X_i \leq t_i) = limit_{t_j \to \infty, j \neq i} F(t_1, t_2, \cdots, t_n)$

(9.4) $X_1, X_2, \cdots X_n$ *are independent if and only if*

$$P(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n) \\ = P(X_1 = x_1)P(X_2 = x_2) \cdots P(X_n = x_n)$$

A collection of random variables is called independent if all of its finite subcollections are independent.

(Theorem 9.2) Let $Y = h(X_1, X_2, \cdots, X_n)$, then

$E(Y) = \sum_{x_n \in A_n} \cdots \cdots \sum_{x_1 \in A_1} h(x_1, x_2, \cdots, x_n) p(x_1, x_2, \cdots, x_n)$.

Example 9.2 on P.397

(9.2) Let

$$p(x,y,z) = k(x^2 + y^2 + yz), x = 0,1,2; y = 2,3; z = 3,4$$

(a) For what values of k is $p(x,y,z)$ a joint probability mass function?

(b) Suppose that, for the value k found in part (a), $p(x,y,z)$ is the joint probability mass function of random variables X, Y, and Z.
Find $P_{Y,Z}(y,z) and\ p_Z(z)$.

(c) Find E(XZ).

Ans: (a) k=1/203; (b)$P_{Y,Z}(y,z) = \frac{1}{203}(3y^2 + 3yz + 5), y = 2,3; z = 3,4$; $p_Z(z) = \frac{15}{203}z + \frac{7}{29}, z = 3,4.$ (c) $E(XZ) = \frac{774}{203} \approx 3.81.$

Example 9.4 on P.402

(9.4) *A system has n components, whose lifetimes are exponential random variables with parameters $\lambda_1, \lambda_2, \cdots, \lambda_n$, respectively. Suppose that the lifetimes of the components are independent random variables, and the system fails as soon as one of its components fails. Find the probability density function and the expected value of the time until the system fails.*

Ans: Let $X_1, X_2, \cdots, X_n$ be the lifetimes of the n components, respectively. Then $P(X_i \leq t) = 1 - e^{\lambda_i t}, t \geq 0$. Let X be the time until the system fails. Then $P(X > t) = P(\min(X_1, X_2, \cdots, X_n) > t)$, and

$$F(t) = 1 - P(X > t) = 1 - \prod_{i=1}^{n} P(X_i > t) = 1 - \prod_{i=1}^{n} e^{-\lambda_i t}$$

$$= 1 - e^{-(\sum_{i=1}^{n} \lambda_i)t}$$

$f(t) = F'(t) = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t}, t \geq 0.$

Joint Probability Density Functions on P.401

(Q9.6) $P\big((X_1, X_2, \cdots, X_n) \in R\big) = \int \cdots \int f(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n$

(Examples and Theorems on P.401-406)

(Definition 9.5) *A Random Sample: we say that n random variables $X_1, X_2, \cdots, X_n$ form a random sample of size n, from a (continuous or discrete) distribution function F, if they are independent and identically distributed. Sample mean (a r.v.) is defined as*

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

9.2 Order Statistics (P.411-417)

Definition 9.6 (P.411)

Let $\{X_1, X_2, \cdots, X_n\}$ be a random sample with pdf *f*, and cdf *F*, for example, $f(x) = 1, x \in [0,1]; and\ F(x) = x, x \in [0,1]$. Let

$$X_{(1)} < X_{(2)} < \cdots < X_{(k)} < \cdots X_{(n)}$$

$$be\ the\ order\ among\ \{X_1, X_2, \cdots, X_n\}.$$

Then, $X_{(k)}$, $1 \leq k \leq n$, is called the *kth* order statistic.

Theorem 9.5 (P.412-413)

Let $\{X_{(1)},\ X_2,\ \cdots, X_{(n)}\}$ be the order statistics of the random sample $\{X_1, X_2, \cdots, X_n\}$ with the common cdf *F*, and pdf *f*, respectively. Then $F_k\ and\ f_k$, the distribution function and probability density function of $X_{(k)}$, respectively, are given as follows: $F_k(x) = P(X_{(k)} \le x)$, that is,

$$F_k(x) = \sum_{i=k}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i},\ -\infty < x < \infty \qquad (9.13)$$

$$f_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1 - F(x)]^{n-k}, -\infty < x < \infty \quad (9.14)$$

Remark: when k=1,

$$F_1(x) = 1 - [1 - F(x)]^n,\ \ f_1(x) = nf(x)[F(x)]^{n-1},$$