6LB. Special Continuous Distributions

- Uniform Distribution
- Exponential Distribution
- Gamma Distribution
- Chi-Square Distribution
- Normal Distribution
- $E(X) = \int_{-\infty}^{\infty} xf(x)dx,\ Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 dx,$
- $\varphi(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx,$
- $E(X) = \varphi'(0),\ Var(X) = \varphi''(0) - [\varphi'(0)]^2$
- Beta Distribution

Uniform Distribution U(a,b), U(0,1)

7.1 on P.279, P.280

Suppose that X is the value of the random point selected from an interval (a, b). *Then X is called a uniform random variable over* (a, b). Then *distribution function* F and *probability density function* f are given below.

$$F(t) = \begin{cases} 0 & t \le a \\ \dfrac{t-a}{b-a} & a < t < b \\ 1 & t \ge b \end{cases}$$

Therefore,

$f(t) = \frac{1}{b-a}$ if a<t<b, and f(t)=0 otherwise. U(0,1) is very common in applications.

X~U(a,b) and X~U(0,1)

- E(X)=$\int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$.
- $\mathrm{E}(X^2) = \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{3}(a^2 + ab + b^2)$.
- Var(X)=$E(X^2) - [E(X)]^2 = \frac{1}{3}(a^2 + ab + b^2) - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$.
- In particular, when $a = 0, b = 1, we\ have$
- $\mathrm{E}(\mathrm{X}) = \frac{1}{2},\ \mathrm{Var}(\mathrm{X}) = \frac{1}{12}$.

7.2 Normal Random Variables $N(\mu, \sigma^2)$

7.2 on P.285

The probability density function *f and its distribution function F* of a normal random variable X with mean μ and variance σ^2 has the following properties.

(a) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2},\ -\infty < x < \infty, denote\ as\ X \sim N(\mu, \sigma^2)$

(b) $F(t) = \int_{-\infty}^{t} f(x)\, dx,\ -\infty < t < \infty$

If $\mu = 0,\ \sigma^2 = 1,\ N(0,1\ )\ is\ callled$ the standard normal distribution.

In this case, we used to use $\varphi\ for\ f\ and\ \Phi\ for\ F$.

$$X \sim N(\mu, \sigma^2) \ \ then \ \ Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

- (a) $X \sim N(\mu, \sigma^2)$ $then$ $Z = \frac{X-\mu}{\sigma} \sim N(0,1)$
- (b) $Z \sim N(0,1)$, $then$ $X = \mu + \sigma Z \sim N(\mu, \sigma^2)$

- If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$, $Var(X) = \sigma^2$, and
- $\varphi(t) = E(e^{tX}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ (moment − generating function)

Figures of Normal Distributions

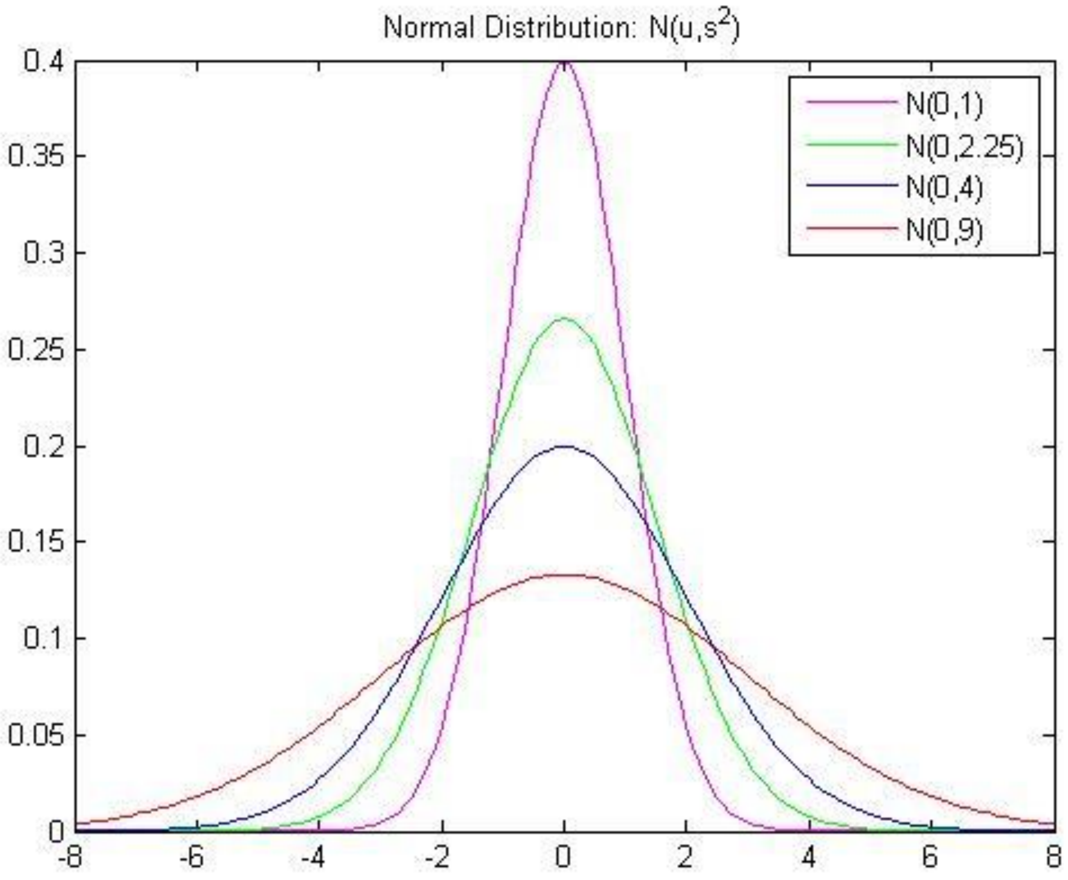
p.d.f. f(x)= $\frac{1}{\sqrt{2\pi s^2}}\ e^{-(x-u)^2\ /2s^2}$, -∞<x<∞

Normal Distribution X~N(u,s^2)

E(X)=u and Var(X)=s^2

Moment-Generating Function φ(t)=$e^{ut+\frac{s^2t^2}{2}}$

```
X=-8:0.1:8;
Y1=normpdf(X,0,1);
Y2=normpdf(X,0,1.5);
Y3=normpdf(X,0,2);
Y4=normpdf(X,0,3);
plot(X,Y1,'m-',X,Y2,'g-',X,Y3,'b-',X,Y4,'r-');
legend('N(0,1)', 'N(0,2.25)', 'N(0,4)', 'N(0,9)')
Title('Normal Distribution: N(u,s^2)')
```



7.3 Exponential Random Variables X: $\lambda e^{-\lambda x}$

7.3 on P.299

The probability density function *f and its distribution function F* of an exponential random variable X with mean $1/\lambda$ and variance $1/\lambda^2$ has the following properties:

(a) $f(x) = \lambda e^{-\lambda x},\ x \geq 0.$

(b) $F(x) = \int_0^x f(t)\, dt = 1 - e^{-\lambda x},\ for\ any\ x \geq 0.$

An alternative form by using $\theta = \frac{1}{\lambda}$ commonly appears in many textbooks; that is $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $x \geq 0$.

Figures of Exponential Distributions

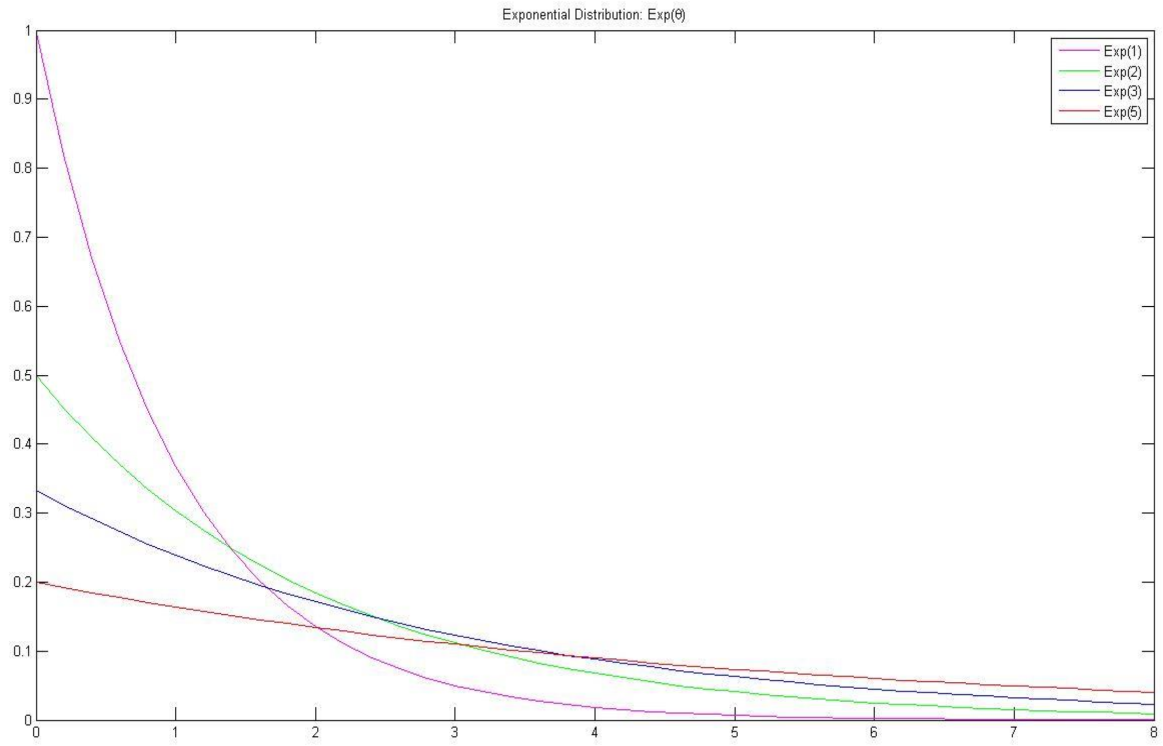
p.d.f. f(x)= $\frac{1}{\theta}\ e^{-x\,/\theta}$, x≥ 0

Exponential Distribution X~Exp(θ)

E(X)=$\theta\ \ and\ \ Var(X) = \theta^2$

Moment-Generating Function φ_X(t)=$\frac{1}{1-\theta t}$

```
X=0:0.2:8;
Y1=exppdf(X,1);
Y2=exppdf(X,2);
Y3=exppdf(X,3);
Y4=exppdf(X,5);
plot(X,Y1,'m-',X,Y2,'g-',X,Y3,'b-',X,Y4,'r-');
legend('Exp(1)', 'Exp(2)', 'Exp(3)', 'Exp(5)')
Title('Exponential Distribution: Exp(\theta)')
```



7.4 Gamma Distributions on P.306

Let $\{N(t): t \geq 0\}$ be a Poisson process, X_1be the time of the first event, and X_n the time between $the\ (n-1)st\ and\ nth\ events$. As explained previously, $\{X_1, X_2, \cdots\}$ is a sequence of identically distributed exponential random variables with mean $1/\lambda$, where λ is the rate of $\{N(t): t \geq 0\}$. For this Poisson process, let X be the time of nth event. Then X is said to have a gamma distribution with parameters (n, λ). Therefore, *exponential* is the time will wait for the first time to occur, and gamma will be the time we will wait for the nth event to occur.

$$F(t) = P(X \leq t) = P(N(t) \geq n) = \sum_{i=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!}$$

Figures of Gamma Distributions

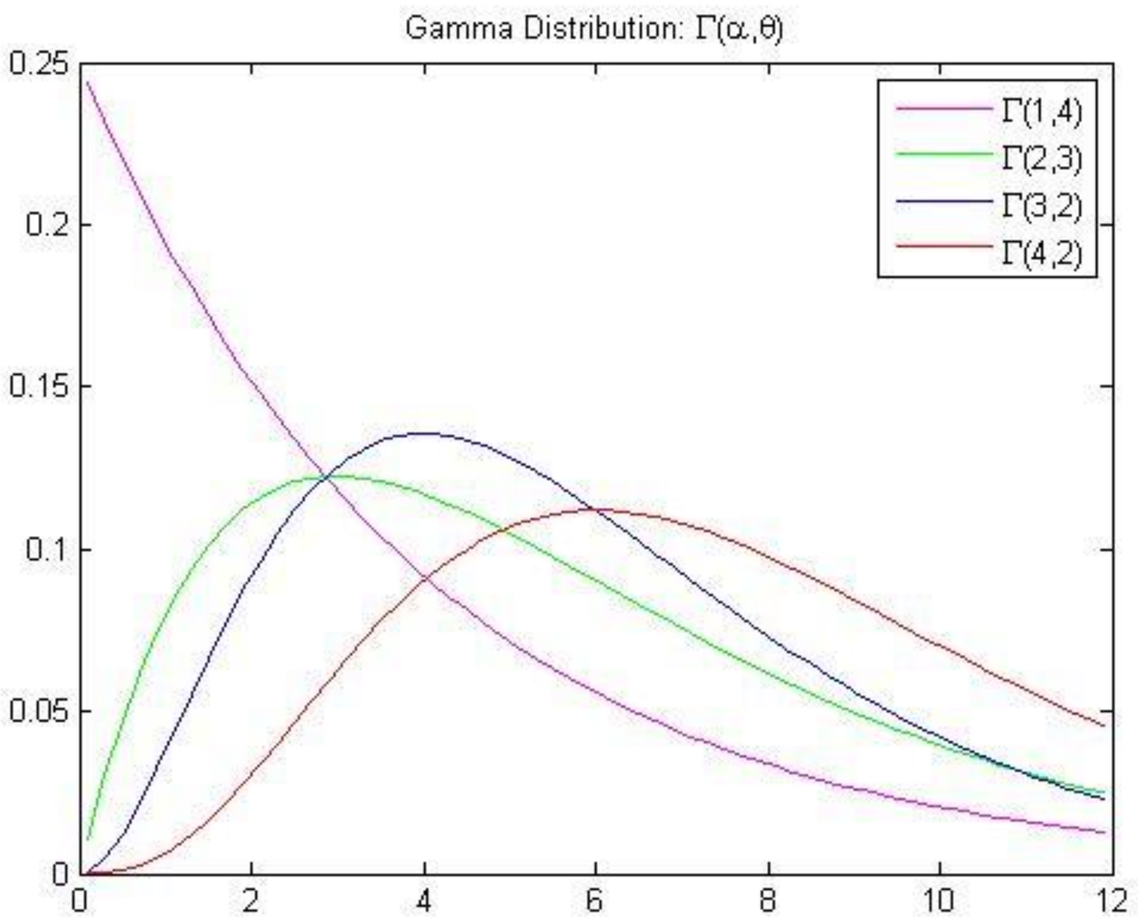
p.d.f. f(x)= $\frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1}e^{-x/\theta}$, 0<x<∞

Gamma Distribution X~$\Gamma(\alpha,\theta)$,

E(X)=$\alpha\theta$, Var(X)=$\alpha\theta^2$

Moment-Generating Function $\varphi_X(t)=\frac{1}{(1-\theta t)^{\alpha}}$

```
X=0.1:0.2:12;
Y1=gampdf(X,1,4);
Y2=gampdf(X,2,3);
Y3=gampdf(X,3,2);
Y4=gampdf(X,4,2);
plot(X,Y1,'m-',X,Y2,'g-',X,Y3,'b-',X,Y4,'r-');
legend('\Gamma(1,4)','\Gamma(2,3)',...
        '\Gamma(3,2)','\Gamma(4,2)')
title(‘Gamma Distribution: \Gamma(\alpha, \theta)’)
```



χ2(r) distribution with r degrees of freedom

- X ~ χ2(r) distribution with r degrees of freedom has the p.d.f.

$$f(x)= \frac{1}{\Gamma(r/2)2^{r/2}}\, x^{(\frac{r}{2})-1}e^{-x\,/2}\,,\ \ 0<x<\infty$$

- where the gamma function is defined as $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$.
- χ^2 distribution is a special case of gamma distribution with parameters $\theta \equiv 2$, and $\alpha = \frac{r}{2}$, where $r\ is\ a\ positive\ integer$.
- Furthermore if X$\sim N(0,1)$, $Y = X^2$ has a Chi-square distribution with 1 degree of freedom, that is, $Y \sim \chi^2(1)$.

Figures of χ^2(r) Distributions

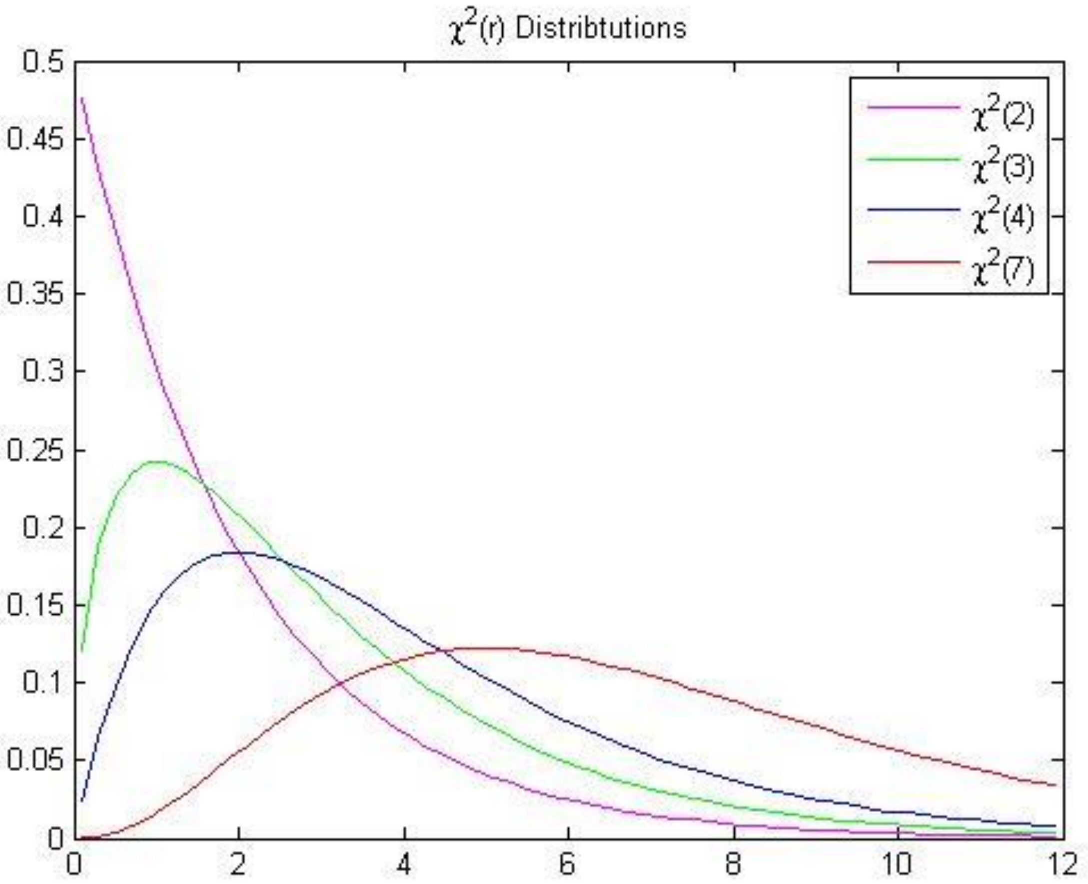
p.d.f. $f(x)=\frac{1}{\Gamma(r/2)2^{r/2}}\, x^{(\frac{r}{2})-1}e^{-x/2}$, $0<x<\infty$

Chi-Square Distribution $X\sim\chi^2(r)$,

E(X)=r, Var(X)=$2r$

Moment-Generating Function $\varphi_X(t)=\frac{1}{(1-2t)^{r/2}}$

```
X=0.1:0.2:12;
Y1=chi2pdf(X,2);
Y2=chi2pdf(X,3);
Y3=chi2pdf(X,4);
Y4=chi2pdf(X,7);
plot(X,Y1,'m-',X,Y2,'g-',X,Y3,'b-',X,Y4,'r-');
legend('\chi^2(2)','\chi^2(3)','\chi^2(4)','\chi^2(7)')
title('\chi^2(r) Distributions')
```



Beta Distributions

p.d.f. $f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0<x<1$

Beta Distribution $X\sim Beta(\alpha,\beta)$,

$E(X)=\alpha/(\alpha+\beta)$, $Var(X)=\alpha\beta/[(\alpha+\beta)^2(\alpha+\beta+1)]$

Moment-Generating Function $M_X(t)$=N/A

```
X=0:0.02:1;
a=2; b=5;  Y1=betapdf(X,a,b);
a=3; b=8;  Y2=betapdf(X,a,b);
a=4; b=4;  Y3=betapdf(X,a,b);
a=9; b=2;  Y4=betapdf(X,a,b);
plot(X,Y1,'r-',X,Y2,'g-',X,Y3,'b-',X,Y4,'m-');
legend('(2,5)','(3,8)','(4,4)','(9,2)',2)
xlabel('x \in [0,1]')
ylabel('Beta p.d.f.')
title('Beta Distribution: Beta(a,b)')
```