A Summary of Special Discrete Distributions

- Discrete Uniform Distribution U({1, 2, 3, …, m})
- Bernoulli trial with parameter p
- Binomial Distribution with parameters n and p
- Poisson Distribution with parameter λ
- Geometric Distribution with parameter p
- Negative Binomial Distribution with parameters r and p
- Hypergeometric Distribution with parameters n, D, N.

Uniform Distribution

When a probability mass function (p.m.f.) is constant on the space, we say that the distribution is uniform over the space. For example, let X take one of the values from S={1, 2, 3, …., m} with the probability $\frac{1}{m}$, then X has a discrete uniform distribution on S = {1, 2, 3, …., m} and its p.m.f. is $p(x) = P(X = x) = \frac{1}{m}, x = 1, 2, \cdots, m$. Then,

$$M(t) = \frac{1}{m}\frac{e^t(1-e^{mt})}{1-e^t},$$

$$E(X) = \frac{m+1}{2}, \qquad \mathrm{Var(X)} = \frac{m^2-1}{12}$$

Bernoulli Trial

A random variable is called Bernoulli (trial) with parameter p if its probability mass function is given by

$$P(X = 1) = p,\ 0 < p < 1,$$
$$P(X = 0) = q = 1 - p,$$

$$M(t) = 1 - p + pe^t,$$
$$E(X) = p,\ Var(X) = p(1 - p).$$

$$E(X) = \sum_{x \in S} xP(X = x) = 0 \times P(X = 0) + 1 \times P(X = 1) = p.$$
$$Var(X) = \sum_{x \in S = \{0,1\}} (x - E(X))^2 P(X = x) = \sum_{x \in S} (x - p)^2 P(X = x),$$

where S$= \{0, 1\}$.

Binomial Distribution X~*b(n,p)*

- If n Bernoulli trials all with probability of success p are performed independently, then X, the number of successes, is called a *binomial with parameters n and p*. The set of possible values of X is {0, 1, 2, …, n}. The probability mass function is given as
- $p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \;\; if\; x = 0, 1, 2, \cdots, n$ (eq. 5.2)
- $M(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = (1-p+pe^t)^n$
- $E(X) = np,\; Var(X) = np(1-p)$
- $E(X) = M'(0) = npe^t(1-p+pe^t)^{n-1}|_{t=0} = np$ (first moment)
- $E(X^2) = M''(0) = n^2p^2 - np^2 + np$ (second moment)
- *Var(X)*=$\mathrm{E}(X^2) - [E(X)]^2 = M''(0) - [M'(0)]^2 = np - np^2 = np(1-p)$

Poisson Random Variable

- Let X be a binomial random variable with parameters (n, p); then
- $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$
- $= \frac{n(n-1)n-2)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i} \rightarrow \frac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, \cdots, \infty$

as $n \rightarrow \infty$, $and\ p\ is\ very\ small\ like\ p \ll 0.1.$

$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, \cdots, \infty$ is called a Poisson p.m.f. and

- $M(t) = e^{\lambda(e^t - 1)}, t > 0$
- $E(X) = \lambda, Var(X) = \lambda$

Geometric Distribution

A geometric distribution has a probability mass function

$$P(X = x) = p(x) = (1 - p)^{x-1}p, \qquad 0 < p < 1, \qquad x = 1, 2, 3, \cdots, \infty$$

$$M(t) = \frac{pe^t}{1-(1-p)e^t}, \ t > 0$$

$$E(X) = \frac{1}{p}, \ Var(X) = \frac{1-p}{p^2}$$

(5.21) From an ordinary deck of 52 cards we draw cards at random, with replacement, and successively until an ace is drawn. What is the probability that at least 10 draws are needed?

Ans: $P(X = n) = \left(\frac{48}{52}\right)^{n-1}\left(\frac{4}{52}\right), n = 1, 2, 3, \cdots, \ P(X \geq 10) = \left(\frac{12}{13}\right)^9 \approx 0.49$

Negative Binomial Random Variables

- Negative binomial r.v.s are generalizations of geometric r.v.s. Suppose that a sequence of independent Bernoulli trials, each with probability of success p, 0<p<1, is performed. Let X be the number of experiments until the rth success occurs. Then X is called negative binomial with parameters (r, p). Its set of possible values is $\{r, r+1, r+2, \cdots\}$ and

$$P(X = x) = \binom{n-1}{r-1} p^r (1-p)^{x-r},\ \ x = r, r+1, r+2, \cdots \qquad (5.5)$$

$$M(t) = \frac{(pe^t)^r}{(1-(1-p)e^t)^r}, \qquad \mathrm{t} > 0$$

$$E(X) = \frac{r}{p},\ Var(X) = \frac{r(1-p)}{p^2}$$

Hypergeometric Random Variable

Suppose that, from a box containing *D* defective and *N-D* nondefective items, n are drawn at random and *without replacement*. Assume that $n = \min(D, N-D)$. Let X be the number of defective items drawn. Then X has the probability mass function

$$p(x) = P(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}, \; x = 0, 1, 2, \cdots, n.$$

Any random variable X with such a p.m.f. is called a hypergeometric r.v.

Note that $\sum_{x=0}^{n} \binom{D}{x}\binom{N-D}{n-x} = \binom{N}{n}$

Expectation and Variance of Hypergeometric Random Variable

$E(X) = \frac{nD}{N}, \; Var(X) = \frac{nD(N-D)}{N^2}\left(1 - \frac{n-1}{N-1}\right).$

Note that if the experiment of drawing n items from a box containing D defective and N-D nondefective items is performed *with replacement,*

Then X is binomial with parameters n *and* $\frac{D}{N}$*, i.,e.,* $X \sim b\left(n, \frac{D}{N}\right)$.

Hence,

$E(X) = \frac{nD}{N}, \; Var(X) = n\frac{D}{N}\left(1 - \frac{D}{N}\right) = \frac{nD(N-D)}{N^2}.$

Reading Examples 5.28 and 5.29

- Practice Review Problems on P.238-240
- Odd numbers only.