5LB. Special Discrete Distributions

- Bernoulli trial with parameter p
- Binomial Distribution with parameters n and p
- Poisson Distribution with parameter λ
- Geometric Distribution with parameter p
- Negative Binomial Distribution with parameters r and p
- Hypergeometric Distribution with parameters n, D, N.

5B. Special Discrete Distributions

- Definition 5.1 A random variable is called Bernoulli with parameter p if its probability mass function is given by

  $P(X = 1) = p,\ 0 < p < 1,$

  $P(X = 0) = q = 1 - p,$

  ~~$0, otherwise.$~~

  $E(X) = \sum_{x \in A} xP(X = x) = 0 \times P(X = 0) + 1 \times P(X = 1) = p.$

  $Var(X) = \sum_{x \in A} (x - E(X))^2 P(X = x) = \sum_{x \in A} (x - p)^2 P(X = x),$

  where $A = \{0, 1\}.$

  $E(X) = p,\ Var(X) = p(1 - p),\ \sigma_X = \sqrt{p(1 - p)}.$

Binomial R.V. = Sum of Indep. Bernoulli Trials

- If n Bernoulli trials all with probability of success p are performed independently, then X, the number of successes, is called a *binomial with parameters n and p*. The set of possible values of X is {0, 1, 2, …, n}. The probability mass function is given as
- $p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \;\; if \; x = 0, 1, 2, \cdots, n$ (eq. 5.2)
- $M(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} = (1-p+pe^t)^n$
- $M'(0) = npe^t(1-p+pe^t)^{n-1}|_{t=0} = np,$
- $M''(0) = n^2p^2 - np^2 + np,$ *Var(X)*$=M''(0) - [M'(0)]^2 = np - np^2$
- $E(X) = np, \; Var(X) = np(1-p), \; \sigma_X = \sqrt{np(1-p)}$

Binomial Distributions

p.m.f. f(x)=$\binom{n}{x}p^x$ (1-p)$^{n-x}$, 0≤ $x \le n$

Binomial Distribution X~b(n,p)

E(X)=np and Var(X)=np(1-p)

Moment-Generating Function M_X(t)=(1-p+pe^t)^n

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p=0.8; n=12; X1=0:n; Y1=binopdf(X1,n,p);
p=0.6; n=12; X2=0:n; Y2=binopdf(X2,n,p);
p=0.6; n=16; X3=0:n; Y3=binopdf(X3,n,p);
p=0.5; n=16; X4=0:n; Y4=binopdf(X4,n,p);
subplot(2,1,1)
bar(X1',[Y1', Y2'])
legend('b(12,0.8)','b(12,0.6)')
subplot(2,1,2)
bar(X3',[Y3', Y4'])
legend('b(16,0.6)','b(16,0.5)')
title('Binomial Distribution: binopdf(X,n,p) \sim b(n,p)')
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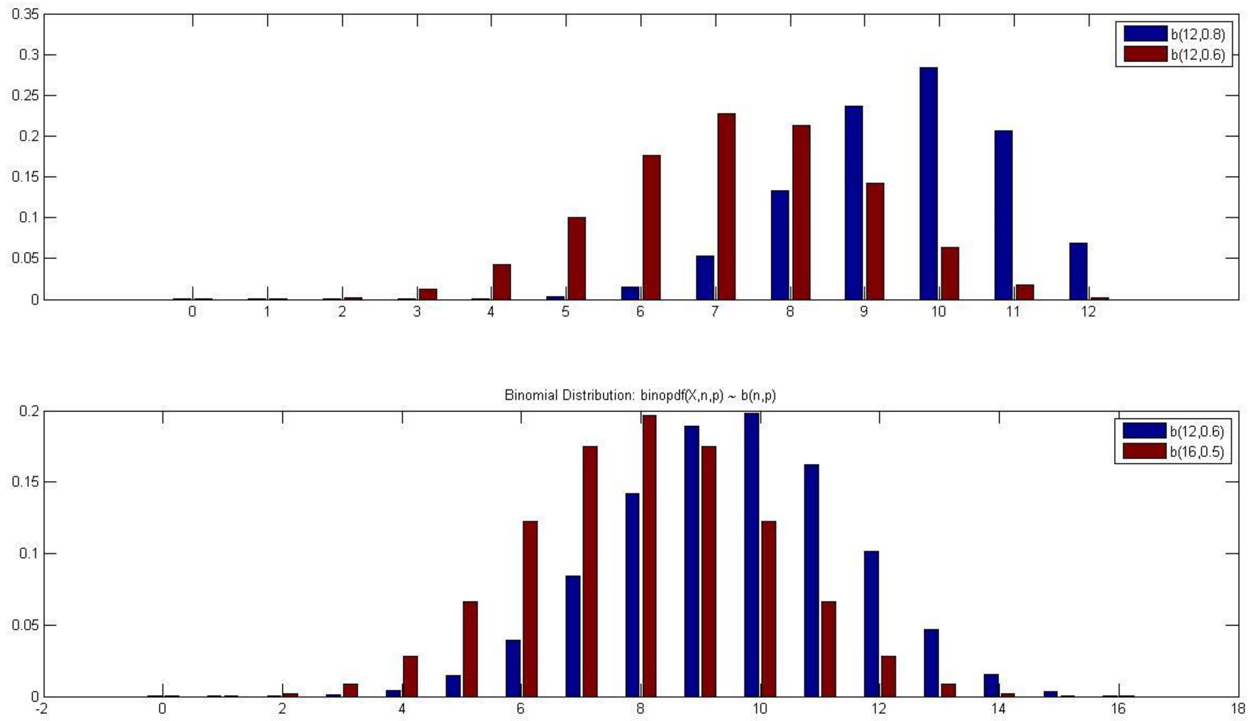

Binomial Random Variable X ~ b(n,p)

- For a binomial distribution as given in equation 5.2 with parameters p and n could be denoted as X ~ b(n,p), where the probability mass function is given below.
- $p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \ \ if\ x = 0, 1, 2, \cdots, n$ (eq. 5.2)
- $\qquad\qquad\qquad\qquad = 0 \ \ elsewhere.$

(Example 5.2) A restaurant serves 8 entrées of fish, 12 of beef, and 10 of poultry. If customers select from these entr$\acute{e}$es, what is the probability that two of the next four customers order fish entrées?

Ans: P(X=2)=$\binom{4}{2}\left(\frac{8}{30}\right)^2\left(1-\frac{8}{30}\right)^{4-2} \simeq 0.23$

Example 5.3 on P.198

- (Example 5.3) In a country hospital 10 babies, of whom 6 were boys, were born last Thursday. What is the probability that the first six births were all boys? Assume that the events that a child born is a girl or a boy are equiprobable.

Ans: Let A be the event that the first six births were all boys and the last four were all girls. Let X be the number of boys among 10 births; then X~ $b(10,1/2)$, then the desired probability is

$$P(A|X=6)=\frac{P(A\ and\ X=6)}{P(X=6)}=\frac{P(A)}{P(X=6)}=\frac{\left(\frac{1}{2}\right)^{10}}{\binom{10}{6}\left(\frac{1}{2}\right)^{6}\left(\frac{1}{2}\right)^{10-6}}=\frac{1}{\binom{10}{6}}\simeq 0.0048$$

Examples 5.4, 5.5, 5.6 on P.198-199

- (5.4) In a small town, out of 12 accidents that occurred in June 1986, four happened on Friday, June 13. Is this a good reason for a superstitious person to argue that Friday the 13th is inauspicious? Yes
- (5.5) Suppose that jury members decide independently and that each with probability p ($0<p<1$) makes the correct decision. If the decision of the majority is final, which is preferable: a 3-person jury or a single juror?

  3-jury if $p>1/2$ or single juror if $p<1/2$
- (5.6) Machuca's favorite bridge hands are those with at least two aces. Determine the number of times he should play in order to have a chance of 90% or more to get at least two favorite hands? Recall that a bridge hand consists of 13 randomly selected cards from an ordinary deck of 52 cards.

Examples 5.7 and 5.8 om P.200

- (5.8) Let p be the probability that a randomly chosen person is pro-life, and let X be the number of persons pro-life in a random sample of size n. Suppose that in a particular random sample of size n persons, k are pro-life. Show that P(X=k) is maximum for $\hat{p}=\frac{k}{n}$. That is, $\hat{p}$ is the value of p that makes the outcome X=k most probable.
- Note that $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$.
- The *mode* of X is defined as the value k of X which maximizes P(X=k).

Example 5.9 (Random Walk) on P.202

- (5.9) One-dimensional random walk

Suppose that a particle is at 0 on the integer number line and suppose that at step 1, the particle will move to 1 with probability p, 0<p<1, and will move to -1 with the probability 1-p. Furthermore, if at step k, the particle is at i, then independently of the previous moves, it will move 1 unit to the right to i+1 with probability p and will move 1 unit to the left to i-1 with probability 1-p. Let X be the position of the particle after n moves. Find the probability mass function of X.

Let Y be the number of times the particle moves one unit to the right. Then $Y \sim b(n, p)$, the set of possible values of X is E={-n, 2-n, 4-n, …, 2n-n}. $P(X = i) = P(2Y - n = i) = P(Y = \frac{n+i}{2})$

5.2 Poisson Random Variables

- Let X be a binomial random variable with parameters (n, p); then
- $P(X = i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{(n-i)!i!}\left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$
- $= \frac{n(n-1)n-2)\cdots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i} \rightarrow \frac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, \cdots, \infty$

as $n \rightarrow \infty$, $and\ p\ is\ very\ small\ like\ p \ll 0.1.$

$\frac{e^{-\lambda}\lambda^i}{i!}, i = 0, 1, 2, \cdots, \infty$ is called a Poisson probability mass function.

Definition 5.3 Poisson Random Variable

- A discrete random variable X with possible values $0, 1, 2, \cdots$ is called Poisson with parameter $\lambda, \lambda > 0, if$
- $P(X = i) = \frac{e^{-\lambda}\lambda^{i}}{i!}, \ i = 0, 1, 2, \cdots$
- $E(X) = \lambda, Var(X) = \lambda, \sigma_X = \sqrt{\lambda}$
- $Extra\ Readings\ from\ P.210 - 222\ about\ Poisson\ Processes$

Geometric Random Variables

- Consider an experiment in which independent Bernoulli trials are performed until the first success occurs. The sample space for such an experiment is
- S={*s, fs, ffs, fffs,* $\cdots, fff\cdots fs, \cdots$}.
- *Suppose that P(s=success)=p, 0<p<1, and P(f=failure)=1-p.*
- Then *P(fffs)=*$(1-p)^3 p$.
- The probability mass function
- $p(x) = (1-p)^{x-1}p, 0 < p < 1, x = 1, 2, 3, \cdots,$ 0 elsewhere.

  is called *geometric.*

Expectation and Variance of Geometric r.v.

The probability mass function

$$P(X = x) = p(x) = (1 - p)^{x-1}p, 0 < p < 1,$$
$$x = 1, 2, 3, \cdots, \text{and } 0 \text{ elsewhere.}$$

is called *geometric.*

$$E(X) = \frac{1}{p}\ , \ Var(X) = \frac{1-p}{p^2}\ , \ \sigma_X = \frac{\sqrt{1-p}}{p}.$$

(5.21) From an ordinary deck of 52 cards we draw cards at random, with replacement, and successively until an ace is drawn. What is the probability that at least 10 draws are needed?

Ans: $P(X = n) = \left(\frac{48}{52}\right)^{n-1}\left(\frac{4}{52}\right), n = 1, 2, 3, \cdots\ ,\ P(X \geq 10) = \left(\frac{12}{13}\right)^9$ ≈0.49

Example 5.22

- A father asks his sons to cut their backyard lawn. Since he does not specify which of the three sons is to do the job, each boy tosses a coin to determine the odd person, who must cut the lawn. In the case that all three get heads or tails, they continue tossing until they reach a decision. Let p be the probability of the heads and $q = 1 - p$, the probability of tails.

(a) Find the probability that they reach a decision in less than n tosses.

(b) If $p = \frac{1}{2}$, what is the *minimum number of tosses* required to reach a decision with probability 0.95?

Negative Binomial Random Variables

- Negative binomial r.v.s are generalizations of geometric r.v.s. Suppose that a sequence of independent Bernoulli trials, each with probability of success p, 0<p<1, is performed. Let X be the number of experiments until the rth success occurs. Then X is called negative binomial. Its set of possible values is $\{r, r+1, r+2, \cdots\}$ and

$$P(X=n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, \; n = r, r+1, r+2, \cdots$$

(5.5) The following probability mass function is called *negative binomial* with parameters (r, p)

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, 0 < p < 1, \qquad x = r, r+1, r+2, \cdots$$

Expectation and Variance of Negative Binomial Random Variable

Negative binomial with parameters (r, p) is given below.

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, 0 < p < 1, \qquad x = r, r+1, r+2, \cdots$$

When $r = 1$, negative binomial becomes geometric binomial.

$E(X) = \frac{r}{p},\ Var(X) = \frac{r(1-p)}{p^2},\ \sigma_X = \frac{\sqrt{r(1-p)}}{p}.$

(5.23) Sharon and Ann play a series of backgammon games until one of them wins five games. Suppose that the games are independent and the probability that Sharon wins a game is 0.58.

(a) Find the probability that the series ends in seven games.

(b) If the series ends in 7 games, what is the probability that Sharon wins?

Example 5.23

(5.23) Sharon and Ann play a series of backgammon games until one of them wins five games. Suppose that the games are independent and the probability that Sharon wins a game is 0.58.

(a) Find the probability that the series ends in seven games.

(b) If the series ends in 7 games, what is the probability that Sharon wins?

Ans: (a) Let X be the number of games until Sharon wins five games, and let Y be the number of games until Ann wins five games. Then X and Y are negative binomial r.v.s with parameters (r, p)=(5,0.58) and (5,0.42), respectively. Thus

P(X=7)+P(Y=7)$= \binom{6}{4}(0.58)^5(0.42)^2 + \binom{6}{4}(0.42)^5(0.58)^2 \approx 0.17 + 0.066 \approx 0.234.$

Answers (a),(b) for Example 5.23

Ans: (a) Let X be the number of games until Sharon wins five games, and let Y be the number of games until Ann wins five games. Then X and Y are negative binomial r.v.s with parameters (r, p)=(5,0.58) and (5,0.42), respectively. Thus

$$P(X=7)+P(Y=7)=\binom{6}{4}(0.58)^5(0.42)^2+\binom{6}{4}(0.42)^5(0.58)^2$$
$$\approx 0.17+0.066\approx 0.234.$$

Ans: (b) Let A be the event that Sharon wins and B be the event that the series ends in seven games. Then the desired probability is

$$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{P(X=7)}{P(X=7)+P(Y=7)}\approx\frac{0.17}{0.234}\approx 0.71$$

Example 5.24 (Attrition Ruin Problem)

(5.24) Two gamblers play a game in which in each play gambler A beats B with probability p, $0 < p < 1$, and loses to B with probability $q = 1 - p$. Suppose that each play results in a forfeiture of $1 for the loser and in no charge for the winner. If player A initially has a dollars and player B has b dollars, what is the probability that B will be ruined?

Ans: Let E_i be the event that, in the first $b + i\ plays$, B loses b times, and let A^* be the event that A wins. Then

$$P(A^*) = \sum_{i=0}^{a-1} P(E_i) = \sum_{i=0}^{a-1} \binom{i+b-1}{b-1} p^b q^i$$

Example 5.25 (Banach Matchbox Problem)

(5.25) A smoking mathematician carries two matchboxes, one in his right pocket and one in his left pocket. Whenever he wants to smoke, he selects a pocket at random and takes a match from the box in that pocket. If each initially contains N matches, what is the probability that when the mathematician for the first time *discovers* that one box is empty (*there are exactly m matches in the other box, m=0, 1, 2, …, N*)?

Every time that the *left* pocket is selected we say that a success has occurred. When the mathematician discovers that the left pocket box is empty, the right one contains m matches iff the (N+1)st success occurs on (N-m)+(N+1)=(2N-m+1)st trial. The probability of this event is

$$\binom{(2N-m+1)-1}{(N+1)-1}\left(\frac{1}{2}\right)^{N+1}\left(\frac{1}{2}\right)^{(2N-m+1)-)N+1)} = \binom{2N-m}{N}\left(\frac{1}{2}\right)^{2N-m+1}$$

Hypergeometric Random Variables

Suppose that, from a box containing *D* defective and *N-D* nondefective items, n are drawn at random and *without replacement*. Assume that $n = \min(D, N - D)$. Let X be the number of defective items drawn. Then X has the probability mass function

$$p(x) = P(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}, \ x = 0, 1, 2, \cdots, n.$$

Any random variable X with such a p.m.f. is called a hypergeometric r.v.

Note that $\sum_{x=0}^{n} \binom{D}{x}\binom{N-D}{n-x} = \binom{N}{n}$

Expectation and Variance of Hypergeometric Random Variable

$E(X) = \frac{nD}{N}, \ Var(X) = \frac{nD(N-D)}{N^2}\left(1 - \frac{n-1}{N-1}\right).$

Note that if the experiment of drawing n items from a box containing D defective and N-D nondefective items is performed *with replacement,*

Then X is binomial with parameters $n \ and \ \frac{D}{N}$*, i.,e.,* $X \sim b\left(n, \frac{D}{N}\right)$.

Hence,

$E(X) = \frac{nD}{N}, \ Var(X) = n\frac{D}{N}\left(1 - \frac{D}{N}\right) = \frac{nD(N-D)}{N^2}.$

Example 5.26

(5.26) In 500 independent calculations a scientist has made 25 errors. If a second scientist checks seven of these calculations randomly, what is the probability that he detects two errors? Assume that the second scientist will definitely find the error of a false calculations?

Ans: Let X be the number of errors found by he second scientist. Then X has a hypergeometric r.v. with parameters N=500, D=25, n=7.

$$P(X=2)=\frac{\binom{25}{2}\binom{500-25}{7-2}}{\binom{500}{7}} \approx 0.04.$$

Example 5.27

(5.27) If a community of $a+b$ potential votes, a are pro-choice and b $(b<a)$ are pro-life. Suppose that a vote is taken to determine the will of the majority with regard to legalizing abortion. If n $(n<b<a)$ random persons of these $a+b$ potential voters do not vote, what is the probability that the pro-life advocates will win?

Let X be the number of those who do not vote and are pro-choice. The pro-life advocates will win iff

$$a-X<b-(n-X).$$

But this is true iff $X>\frac{a-b+n}{2}$. Since X is hypergeometric,

$P\left(X>\frac{a-b+n}{2}\right)=\sum_{i=[\frac{a-b+n}{2}]}^{n}P(X=i)=\sum_{[\frac{a-b+n}{2}]}^{n}\frac{\binom{a}{i}\binom{b}{n-i}}{\binom{a+b}{n}}$, $[*]$ is the floor of $*$

Reading Examples 5.28 and 5.29

- Practice Review Problems on P.238-240
- Odd numbers only.