3. Conditional Probability and Independence

- 1. Conditional Probability
- 2. The Multiplication Rule
- 3. Law of Total Probability
- 4. Bayes' Formula
- 5. Independence
- Introduction: Suppose that 140 CS freshmen took both discrete math and calculus in the same class, 70% of the students passed calculus, 55% passed discrete math, and 45% passed both. If a randomly selected freshman is found to have passed calculus, what is the probability that he or she has also passed discrete math?
- Let A, B be the events that a randomly selected student who passed discrete math and calculus, respectively. Then
- P(A)=0.55, P(B)=0.70, and P(A∩B)=0.45. P(A|B)=$\frac{P(A \cap B)}{P(B)}$=$\frac{0.45}{0.70}$

3.1 Conditional Probability (P.88)

- (3.1 P.88) If P(B)>0, the conditional probability of A given B, denoted by P(A|B), is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

- (3.1, P.88) In a certain region of Russia, the probability that a person lives at least 80 years is 0.75, and the probability that he or she lives at least 90 years is 0.63. What is the probability that a randomly selected 80-year-old person from this region will survive to become 90?
- Ans: $P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.63}{0.75} \approx 0.84.$
- (3.2, P.88) From the set of all families with two children, a family is selected at random and is found to have a girl. What is the probability that the other child of the family is a girl? Assume that in a two-child family all sex distributions are equally probable. Answer: (1/4)/(3/4)=1/3
- (3.3, P.89) From the set of all families with two children, a child is selected at random and is found to be a girl. What is the probability that the second child of this girl's family is also a girl? Assume that in a two-child family all sex distributions are equally probable. Answer: (1/4)/(1/2)=1/2
- (3.4, P.89) An English class consists of 10 Koreans, 5 Italians, and 15 Hispanics. A paper is found belonging to one of the students of this class. If the name on the paper is not Korean, what is the probability that it is Italian? Assume that names completely identify ethnic groups. Answer: (5/30)/(20/30)=5/20
- (3.5, P.90) Cromwell's Rule: $P(B) = P(BA) + P(BA^c)$ with P(B)>0.

3.1 Conditional Probability

- (Theorem 3.1) *Let S be the sample space of an experiment, and let B be an event of S with P(B)>0. Then*
- $(a)\ P(A|B) \geq 0 for\ any\ event\ A\ of\ S.$
- $(b)\ P(S|B) = 1.$
- (c) *If* $A_1, A_2, \cdots, A_n$*, is a sequence of mutually exclusive events, then*

$$P(\cup_{i=1}^{\infty} A_i\ |B) = \sum_{i=1}^{\infty} P(A_i|B)$$

- Reduction of Sample Space (P.91)
- Let B be an event of a sample space S with P(B)>0. For a subset A of B, define Q(A)=P(A|B). Then Q is a function from the set of subsets of B to [0,1]. Clearly, Q(A)≥ 0, Q(B)=P(B|B)=1 and if *If* $A_1, A_2, \cdots$*, is a sequence of mutually exclusive events of B, then*

$$Q(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} A_i\ |B) = \sum_{i=1}^{\infty} P(A_i|B) = \sum_{i=1}^{\infty} Q(A_i)$$

Reduction of Sample Space (P.91-93)

- Let B be an event of a sample space S with P(B)>0. For a subset A of B, define Q(A)=P(A|B). Then Q is a function from the set of subsets of B to [0,1]. Clearly, Q(A)≥ 0, Q(B)=P(B|B)=1 and if *If* $A_1, A_2, \cdots$, *is a sequence of mutually exclusive events of B, then*

$$Q(\cup_{i=1}^{\infty} A_i) = P(\cup_{i=1}^{\infty} A_i \mid B) = \sum_{i=1}^{\infty} P(A_i \mid B) = \sum_{i=1}^{\infty} Q(A_i)$$

- (3.7, P.91) A child mixes 10 good and three dead batteries. To find the dead batteries, his father tests them one-by-one and without replacement. If the first four batteries tested are all good, what is the probability that the fifth one is dead? Ans: 3/9

- ~~(3.8, P.91) A farmer decides to test four fertilizers for his soybean fields. He buys 32 bags of fertilizers, eight bags from each kind (type), and tries them randomly on 32 plots, eight plots from each of fields A, B, C, and D, one bag per plot. If from type I fertilizer one bag is tried on field A and three on field B, what is the probability that two bags of type I fertilizer are tried on field D?~~

- ~~Ans:~~ $\binom{4}{2}\binom{12}{6}/\binom{16}{8} \approx 0.43$

Exercises A from P.93

(A1, P.93) Suppose that 15% of the population of a country are unemployed women, and a total of 25% are unemployed. What percent of the unemployed are women? Ans: $P(W|Unemployed)=0.25/0.15=0.60$.

(A3, P.93)In a technical college all students are required to take calculus and physics. Statistics show that 32% of the students of this college get A's in calculus, and 20% of them get A's in both calculus and physics. Alice, a randomly selected student of this college, has passed calculus with an A. What is the probability that she also got an A in physics? Ans: 0.20/0.32.

(A4, P.93) Suppose that two fair dice have been tossed and the total of their top faces is found to be divisible by 5. What is the probability that both of them have landed 5?

Ans: $E=\{(x,y)|x+y=5 \text{ or } x+y=10, 1\leq x,y\leq 6\}$, $F=\{(5,5)\}$. $P(F|E)= 1/7$.

Exercises A from P.94

(A8, P.94) In throwing two fair dice, what is the probability of a sum of 5 if they land on different numbers? Ans: 4/30.

(A10, P.94) From 100 cards numbered 00, 01, …, 99, one card is drawn. Suppose that α and β are the sum and the product, respectively, of the digits of the card selected. Calculate P({$\alpha = i | \beta = 0$}), $0 \leq i \leq 18$.

Ans: P({$\beta = d_1 \times d_2 = 0$})=0.19. P({$\alpha = j | \beta = 0$}) $= 0, 10 \leq j \leq 18$.

(A9, P.94) In a small lake, it is estimated that there are approximately 105 fish, of which 40 are trout and 65 are carp. A fisherman caught 8 fish; what is the probability that exactly two of them are trout if we know that at least three of them are not?

Ans: [C(40,2)C(65,6)/C(105,8)] / [1-$\sum_{i=0}^{2} C(40,8-i)C(65,i)/C(105,8)$]

Exercises B from P.95

(B20, P.95) A number is selected at random from the set of {1, 2, 3, …, 10000} and is observed that to be odd. What is the probability that it is

(a) divisible by 3; Ans: 1667/5000.

(b) divisible by neither 3 nor 5? Ans: $1 - \frac{1667}{5000} - \frac{1000}{5000} + \frac{333}{5000} = 0.5332.$

(B22, P.95) A big urn contains 1000 red chips, numbered 1 through 1000, and 1750 blue chips, numbered 1 through 1750. A chip is removed at random, and its number is found to be divisible by 3. What is the probability that its number is also divisible by 5.

Ans: (66+116)/(333+583)=182/916.

3.2 Multiplication Rule (P.97)

- (Definition) If P(B)>0, the conditional probability of A given B, denoted by P(A|B), is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}, \text{ or } P(A \cap B) = P(A|B)P(B)$$

- Similarly, if P(A)>0, we have

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(BA)}{P(A)}, \text{ or } P(A \cap B) = P(B|A)P(A)$$

(3.10, P.97) Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests ? (Ans: $P(D_1 D_2) = (\frac{2}{7}) \times \left(\frac{1}{6}\right) = (\frac{1}{21})$

(Theorem 3.2) (The multiplication rule) If $P(A_1 A_2 \cdots A_{n-1})>0$, then

$$P(A_1 A_2 \cdots A_n) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2) \cdots P(A_n | A_1 A_2 \cdots A_{n-1}).$$

Examples 3.11 and 3.12 (P.98)

- (3.11, P.98) A consulting firm is awarded 43% of the contracts it bids on. Suppose that Andy works for a division of the firm that gets to do 15% of the projects contracted for. If Andy directs 35% of the projects submitted to his division, what percentage of all bids submitted by the firm will result in contracts for projects directed by Andy?
- Ans: $P(A_1A_2A_3)$=$P(A_1)P(A_2|A_1)P(A_3|A_1A_2)$=(0.43)(0.15)(0.35)
- (3.12, P.98) Suppose that five good and two defective fuses have been mixed up. To find the defective ones, we tset one by one, at random and without replacement. What is the probability that we find both of the defective fuses *in exactly three tests*?
- Ans: $P(G_1D_1D_2 \cup D_1G_2D_3)$=$\left(\frac{5}{7}\right)\left(\frac{2}{6}\right)\left(\frac{1}{5}\right)+\left(\frac{2}{7}\right)\left(\frac{5}{6}\right)\left(\frac{1}{5}\right)=\left(\frac{2}{21}\right)\approx 0.09524$

3.3 Law of Total Probability

(Theorem 3.3) Let B be an event with $P(B) > 0$ and $P(B^c) > 0.$ Then for any event A,

- $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

(3.13, P.101) An insurance company rents 35% of the cars for its customers from agency I and 65% from agency II. If 8% of the cars of agency I and 5% of the cars of agency II break down during the rental periods, what is the probability that a car rented by this insurance company breaks down?

- Ans: P(A)=P(A|I)P(I)+P(A|II)P(II)=(0.08)(0.35)+(0.05)(0.65)=0.0605

Example 3.14 on P.102

(3.14, P.102) In a trial, the judge is 65% sure that Susan has committed a crime. Julie and Robert are two witness who know whether Susan is innocent or guilty. However, Robert is Susan's friend and will lie with probability 0.25 if Susan is guilty. He will tell the truth if she is innocent. Julie is Susan's enemy and will lie with probability 0.30 if Susan is innocent. She will tell the truth if Susan is guilty. What is the probability that, in the course of trial, Robert and Julie will give conflict testimony?

- Solution: Let I be the event that Susan is innocent. Let C be the event that Robert and Julie will give conflicting testimony. By the law of total probability,
- $P(C)=P(C|I)P(I)+P(C|I')P(I')=(0.3)(0.35)+(0.25)(0.65)=0.2675.$

3.15 Gambler's Ruin Problem on P102 (Hard)

- Two gamblers play the game of "heads or tails," in which each time a fair coin lands heads up player A wins \$1 from B, and each time it lands tails up, player B wins \$1 from A. Suppose that player A initially has a dollars and player B has b dollars. If they continue to play this game successively, what is the probability that (i) A will be ruined; (ii) the game goes forever with nobody winning?
- *Let E be the event that A will be ruined if he or she starts with* i *dollars, and let* $p_i = P(E)$. *We want to calculate* $p_{a.}$ *Define F to be the event that A wins the first game. Then,*
- $P(E) = P(E|F)P(F) + F(E|F^c)P(F^c)$, *or* $p_i = p_{i+1} \cdot \frac{1}{2} + p_{i-1} \cdot \frac{1}{2}$.

A Partition of Sample Space S (P.104)

(3.2) Let $\{B_1, B_2, \cdots, B_n\}$ be a set of nonempty subsets of the sample space S of an experiment. If the events $B_1, B_2, \cdots, B_n$ are mutually exclusive and $\bigcup_{i=1}^{n} B_i$=S, the set $\{B_1, B_2, \cdots, B_n\}$ is called a partition of S.

(Theorem 3.4, Law of Total Probability)

If $\{B_1, B_2, \cdots, B_n\}$ is a partition of the sample space of an experiment and $P(B_i)$>0 for $i = 1, 2, \cdots, n$, then for any set A of S,

$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Example 3.16 on P.105

- (3.16, P.105) Suppose that 80% of the seniors, 70% of the juniors, 50% of the sophomores and 30% or the freshmen of a college use the library of their campus frequently. If 30% of all students are freshmen, 25% are sophomores, 25% are juniors, and 20% are seniors, what percent of all students use the library frequently?
- Solution: Let U be the event that a randomly selected student is using the library frequently. Let F, O, J, E be the events that he or she is a freshman, sophomore, junior, or senior, respectively. Then, {F, O, J, E} is a partition of the sample space S. Thus

$$P(U)=P(U|F)P(F)+P(U|O)P(O)+P(U|J)P(J)+P(U|E)P(E)$$

$$=(0.30)(0.30)+(0.50)(0.25)+(0.70)(0.25)+(0.80)(0.20)=0.55$$

Examples 3.17 and 3.18 on p.106

- (3.17, 106) Suppose that the only parasite living in an aquatic habitat is a single-celled organism which after a second, with equal probabilities, either splits into two organisms, remains as is, or dies. Suppose that in subsequent seconds, all the living parasites in the habitat follow the same behavior as the original parasite, independently of each other. What is the probability that eventually the aquatic habitat will be clean with no parasites?
- Solution: The aquatic habitat will eventually be clean if ultimately the parasite population dies out. Let A be the event that this happens. Let T, R, D be the events that, after a second, the parasite splits into two organisms, remains the same, and dies, respectively. By the law of total probability,
- $P(A)=P(A|T)P(T)+P(A|R)P(R)+P(A|D)P(D)$
- $=[P(A)]^2\times(1/3)+P(A)\times(1/3)+1\times(1/3)$, thus, $P(A)=1$.

Solution for Example 3.18 on P.107

(3.18, P.107) An urn contains 10 white and 12 red chips. Two chips are drawn at random and, without looking at their colors, are discarded. What is the probability that a third chip drawn is red?

- Solution: For $i \geq 1$, let R_i be the event that the ith chip drawn is red, and W_i be the event that it is white. Then
- $P(R_3)$=P($R_3 | R_2W_1$)P(R_2W_1)+ P($R_3 | W_2R_1$)P(W_2R_1)

  +P($R_3 | R_2R_1$)P(R_2R_1) P($R_3 | W_2W_1$)P(W_2W_1)

  $=\left(\frac{11}{20}\right) \times \left(\frac{20}{77}\right) + \left(\frac{11}{20}\right) \times \left(\frac{20}{77}\right) + \left(\frac{10}{20}\right) \times \left(\frac{22}{77}\right) + \left(\frac{12}{20}\right) \times \left(\frac{15}{77}\right) = \frac{12}{22}$

Note that: The two discarded chips provide no information.

3.4 Bayes' Formula (P.111)

- (Theorem 3.5) (Bayes' Formula)
- Let $\{B_1, B_2, \cdots, B_n\}$ be a partition of the sample space S of an experiment. If for $i = 1, 2, \cdots, n, P(B_i)$>0, then for any event A of S with P(A)>0,
- $P(B_k|A) = \frac{P(A|B_k)P(B_k)}{\sum_{i=1}^{n} P(A|B_i)P(B_i)}$

- Examples 3.19, 3.20, 3.21, 3.22, 3.23
- Exercises

Example 3.19 on P.113

- (3.19, P.113) In a study conducted three years ago, 82% of the people in a randomly selected sample were found to have good financial credit ratings, while the remaining 18% were found to have bad financial credit ratings. Current records of the people from that sample show that 30% of those with bad credit ratings have since improved their ratings to good, while 15% of those with good credit ratings have since changed to having a bad credit rating. What percentage of people with good credit ratings now had bad ratings three years ago?
- Solution: Let G be the event that a randomly selected person from the sample has good rating now, and let B be the event that he or she had a bad credit rating three years ago. The solution $P(B|G)=0.072$ can be found by Bayes' formula with $P(G|B)=0.30$, $P(B)=0.18$, $P(B')=0.82$, $P(G|B')=0.85$.

Example 3.23 (P.115)

(3.23, P.115) A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colors being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

- Solution: Let BB, BR, RR be events that the discarded balls are blue and blue, blue and red, and red and red, respectively, and let R be the event that the third ball drawn is red. Since {BB, BR, RR} is a partition of the sample space, Bayes' formula can be used to calculate P(BB|R).
- P(BB|R)= $\frac{P(R|BB)P(BB)}{P(R|BB)P(BB)+P(R|BR)P(BR)+P(R|RR)P(RR)} \approx 0.46$

3.5 Independence (P.120)

- (Definition 3.3, P.120) Two events A and B are called independent if $P(A \cap B)=P(AB)=P(A)P(B)$. If two events are not independent, they are called dependent.
- (3.24, P.120) In the experiment of tossing a fair coin twice, let A and B be the events of getting heads on the first and second tosses, respectively. Then A and Bare independent because $P(AB)=(1/4)$, $P(A)=P(B)=1/2$, and $P(AB)=P(A)P(B)$.
- (3.6, P.122) If A and B are independent, then A and B' are independent as well.
- (Corollary, P.122) If A and B are independent, then A' and B' are independent as well.

Independence of more than 2 events (P.122)

- The events A, B, C are called independent if

P(AB)=P(A)P(B), P(AC)=P(A)P(C), P(BC)=P(B)P(C), P(ABC)=P(A)P(B)P(C)

If A, B, C are independent events, we say that {A, B, C} are an independent set of events.

The set of events $\{A_1, A_2, \cdots, A_n\}$ is called independent if for every subset of $\{A_{i_1}, A_{i_2}, \cdots, A_{i_k}\}, k \geq 2$, of $\{A_1, A_2, \cdots, A_n\}$,

$$P\left(A_{i_1} A_{i_2} \cdots A_{i_k}\right) = P\left(A_{i_1}\right) P\left(A_{i_2}\right) \cdots P\left(A_{i_k}\right)$$

- (3.29, P.122) Let an experiment consist of throwing a die twice. Let A be the event that in the second throw the die lands 1, 2, or 5; B be the event that in the second throw it lands 4, 5, or 6; and C be the event that the sum of the two outcomes is 9. Then P(A)=P(B)=1/2, P(C)=1/9, and P(ABC)=1/36= P(A)P(B)P(C); however, P(AB)≠P(A)P(B), P(AC)≠ $P(A)P(C)$, P(BC)≠P(B)P(C)

Example 3.37 Tossing n+1 time and n times

- Adam tosses a fair coin n+1 time, Andrew tosses the same coin n times. What is the probability that Adam gets more heads than Andrew?
- Solution: Let H_1, H_2 (T_1, T_2) be the number of heads (tails) obtained by Adam and Andrew, respectively. Since the coin is fair,
- $P(H_1 > H_2)$=$P(T_1 > T_2)$, but
- $P(T_1 > T_2)$= $P(n+1-H_1 > n-H_2)$=$P(H_1 \leq H_2)$, thus
- $P(H_1 > H_2)$=$P(H_1 \leq H_2)$, moreover, $P(H_1 > H_2)$+$P(H_1 \leq H_2) = 1$,

which implies $P(H_1 > H_2)$=$P(H_1 \leq H_2) = 1/2.$

On the other hand, $P(H_1 > H_2)$= $\sum_{i=0}^{n} P(H_1 > H_2 | H_2{=}i) P(H_2 = i)$

$= \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} P(H_1 = \mathrm{j}) \mathrm{P}(H_2 = i) = \frac{1}{2^{2n+1}} \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} \binom{n}{i}$, therefore, we have

$$\sum_{i=0}^{n} \sum_{j=i+1}^{n+1} \binom{n+1}{j} \binom{n}{i} = 2^{2n}$$

3R. Review Problems P.139-143

1. Two persons arrive at a train station, independently of each other, at random time between 1:00 pm and 1:30 pm. What is the probability that one will arrive between 1:00 pm and 1:12 pm, and the other between 1:17 pm and 1:30 pm? (Ans: $\frac{12}{30} \times \frac{13}{30} + \frac{12}{30} \times \frac{13}{30} = \frac{104}{300} \approx 0.347$)
3. A polygraph operator detects innocent suspects as being guilty of the time with the probability 3%. If during a crime investigation six innocent suspects are examined by the operator, what is the probability that at least one of them is detected as guilty? (Ans: $1-(1-0.03)^6 = 1-(0.97)^6 \approx 0.167$)
5. In statistical surveys where individuals are selected randomly and are asked questions, experience has shown that only 48% of those under 25 years of age, 67% between 25 and 50, and 89% above 50 will respond. A social scientist is about to send a questionnaire to a group of randomly selected people. If 30% of the population are younger than 25 and 17% are older than 50, what percent will answer her questionnaire?

   Ans: 0.48*0.30+0.67*0.53+0.89*0.17=0.6504

3R. Review Problems P.139-143

13. From an ordinary deck of 52 cards, 10 cards are drawn at random. If exactly four them are hearts, what is the probability of at least one spade being among them? Ans: $1-\frac{\binom{13}{4}\binom{26}{6}}{\binom{13}{4}\binom{39}{6}} \approx 0.929$

15. Suppose that 10 dice are thrown and we are told that among them at least one has landed 6. What is the probability that there are two or more sixes? Ans: $[1-\left(\frac{5}{6}\right)^{10} - 10\left(\frac{5}{6}\right)^{9}\left(\frac{1}{6}\right)^{1}] / [1-\left(\frac{5}{6}\right)^{10}] \approx 0.615$

23. A fair coin is tossed. If the outcome is a head, a red hat is placed on Lorna's head. If it is a tail, a blue hat is placed on her head. Lorna cannot see the hat. She is asked to guess the color of her hat. Is there a strategy that maximizes Lorna's chances of guessing correctly?

    Ans: P(C)=P(C|Red)P(Red)+P(C|Blue)P(Blue)=$c*\frac{1}{2}+(1-c)*\frac{1}{2}=\frac{1}{2}$