2. Combinatorial Methods (P.45)

- We study a few rules that enable us to count systematically. Combinatorial analysis deals with methods of counting.
- Counting Principles
- Permutations
- Combinations
- Stirling's Formula

2.2 Counting Principles

- (2.1) *Counting Principle*: If the set E contains n elements and F contains m elements, there are nm ways in which we can choose a pair of (e,f), where *e belongs to E* and *f belongs to F.*
- *Generalized Counting Principle:* Let $E_1, E_2, \ldots, E_k$ be sets with $n_1, n_2, \ldots, n_k$ elements, respectively. Then there are $n_1 \times n_2 \times \cdots \times n_k$ ways in which we can choose a k-tuple $(e_1, e_2, \ldots, e_k)$ such that $e_j \in E_j$, $1 \leq j \leq k$.
- Example 1. How many outcomes are there if we toss 3 coins? (2^3)
- (2.1) How many outcomes are there if we throw 5 dice? (6^5)
- Theorem: A set with n elements has 2^n subsets.

2.2 Counting Principles

- (2.2) In tossing four fair dice, what is the probability of at least one 3? (1-$(\frac{5}{6})^4$) $= 1 - \frac{625}{1296} = \frac{671}{1296}$ ≈0.52)
- (2.3) Sandy wants to give her son, Ryan, 14 different baseball cards within a 7-day period. If Sandy gives Ryan cards no more than once a day, in how many ways can this be done? (Ans: 7^{14})
- (2.6) Cindy has invited n friends to her party. If they all attend, and each one shakes with everyone else at the party exactly once, what is the number of handshakes)? (Ans: C(n+1,2) since there n+1 people)

2.2 Counting Principles

- (2.8) (Standard Birthday Problem) What is the probability that at least two students of a class of size n have the same birthday? Compute the numerical values of such probabilities for $n = 23, \quad 30, \quad 50, \quad 60.$

  $Ans: 0.507, 0.706, 0.970, 0.995.$
- (2.10) A restaurant advertises that it offers over 1000 varieties of pizza. If, at the restaurant, it is possible to have on a pizza any combination of pepperoni, mushrooms, sausage, green peppers, onions, anchovies, salami, bacon, olives, and ground beef, is the restaurant advertisement true?
- (2.11) Bill and John keep playing chess until one of them wins two games in a row or three games altogether. In what percent of all possible cases does the game end because Bill wins three games without winning two in a row?

2.2 Counting Principles (P.51-52)

- (2.12, P.51) Mark has \$4. He decides to bet \$1 on the flip of a fair coin four times. What is the probability that (a) he breaks even; (b) he wins money?
- (A14) How many n by m arrays (matrices) with entries 0 or 1?
- (Aoo) How many divisors does 10 has? (10=2×5)
- (A15, P.52) How many divisors does 55125 have? ($55125=3^2 5^3 7^2$)
- (Aoo) How many relatively prime numbers does an n have , that is, the numbers of k in {1,2,3 ...,n} such that gcd(k,n)=1? (ans: $\varphi(n)$)

2.3 Permutations (P.55)

- (2.1) An ordered arrangement of r objects from a set A containing n objects ($0<r\leq n$) is called an r-element permutation of A, or a permutation of the elements of A taken r at a time. The number of r-element permutations of a set containing n objects is denoted by

$$ {}_nP_r = P_r^n = P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!} = [C(n,r)r!]; \quad P_n^n = n! $$

- (2.13, P.57) *Alice, Becky, Cindy must be scheduled to meet their thesis advisor. In how many different orders can this be done?*
- (2.oo) Let an vehicle license plate number in a certain region is composed of two English letters followed by 4 digit numbers. How many possible license plate numbers can be issued in that region?

  Ans: $26\times 26 \times 10^4$=5760000=5.76 million

2.3 Permutations (P.59)

(A8, P.59) Let A be the set of all sequences of 0's, 1's, and 2's of length 12.

(a) How many elements are there in A? (Ans: $3^{12} = 531441$)

(b) How many elements of A have exactly six 0's and six 1's? (Ans: $\frac{12!}{6!6!}$=924)

(c) How many elements of A have exactly three 0's, four 1's, and five 2's?

(Ans: $\frac{12!}{3!4!5!}$ =27720)

(2.15, P.57) If five boys and five girls sit in a row in a random order, what is the probability that no two children of the same sex sit together?

(Ans: $\frac{2\times 5!5!}{10!} \approx 0.008$)

2.3 Permutations (P.58)

- (2.4, P.58) *The number of distinguishable permutations of n objects of k different types, where n_1 are alike, n_2 are alike, …, n_k are alike and $n=n_1+n_2+\ldots+n_k$, is* $\frac{n!}{n_1!\times n_2!\ldots\times n_k!}$
- (2.16) How many different 10-letter codes can be made using three a's, four b's, and three c's? (Ans: $\frac{10!}{3!\times 4!\times 3!}$=4200)
- (2.17) In how many ways can we paint 11 offices so that four of them will be painted green, three yellow, two purple, and the remaining two pink? (Ans: $\frac{11!}{4!\times 3!\times 2!\times 2!}$=69300)

2.3 Permutations

- (A4, P.59) How many messages can be sent by five dashes and three dots? (Ans: $\frac{8!}{5!\times 3!}$=56)
- (2.18, P.58) A fair coin is flipped 10 times. What is the probability of obtaining exactly three heads? (Ans: $(\frac{10!}{3!\times 7!})/2^{10} \approx 0.12$)
- (Aoo) A fair die is tossed eight times. What is the probability of exactly two 3's, three 1's, and three 6's?

  (Ans: $(\frac{8!}{2!\times 3!\times 3!})/6^{8} \approx 0.0005$)

2.4 Combinations (P.62)

(2.2) An unordered arrangement of r objects from a set A containing n objects ($r \leq n$) is called *an r-element combination* of A, or a combination of the elements of A taken r at a time. The number of *r-element combinations of n* objects can be denoted as

$${}_nC_r = C_r^n = \mathrm{C(n, r)} = \binom{n}{r} = \frac{n!}{(n-r)!r!}$$

(2.19, P.63) In how many ways can two Calculus and three Linguistics books be selected from eight Calculus and six Linguistics books?

Answer: $\binom{8}{2}\binom{6}{3}$=560

2.4 Combinations

(2.20, P.64) A random sample of 45 instructors from different state universities were selected randomly and asked whether they are happy with their teaching loads. The responses of 32 were negative. If Drs. Smith, Brown, and Jones were among those questioned, what is the probability that all three of them gave negative responses?

Ans: $\frac{\binom{45-3}{32-3}}{\binom{45}{32}} \approx 0.35$

(2.21, P.64) In a small town, 11 of 25 school teachers are against abortion, eight are for abortion, and the rest are indifferent. A random sample of five school teachers is selected for an interview. What is the probability that (a) all of them are for abortion; (b) all of them have the same opinion?

Ans: (a) 0.0011; (b) 0.0099

2.4 Combinations (P.66)

(2.25, P.66) Show that the number of different ways n identical objects can be distributed to k people $(k \leq n)$ is

- $\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$, some people could receive nothing.
- $\binom{n-1}{k-1} = \binom{n-1}{n-k}$ if each people receives at least one object.

(2.oo) Verify (n, k)=(3, 2), (4, 2), (5, 3), (6, 3) in Example 4.

2.4 Combinations (P.67)

(2.26, P.67) Let n be a positive integer, and consider the integer equation $x_1 + x_2 + \cdots + x_k = n$.

- (a) How many distinct positive integer solutions does the equation have?
- Ans: $\binom{n-1}{n-k}$=$\binom{n-1}{k-1}$
- (b) How many distinct nonnegative integer solutions does the equation have?
- Dividing n identical objects into k cells, $\binom{n+k-1}{n}$=$\binom{n+k-1}{k-1}$

2.4 Combinations (P.69)

(2.5, P.69) Binomial Expansion: $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$ for $n \geq 0$

(2.30, P.70) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

- $\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$
- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

(2.31, P.70) $\sum_{i=1}^{n} \binom{n}{i} i = n \times 2^{n-1}$

(2.32, P.70) $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \times \binom{n}{n-i}$

2.4 Combinations

- Multinomial Expansion: In the expansion of $(x_1 + x_2 + \cdots + x_k)^n$, the coefficient of the term $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_k^{n_k}$, $n_1 + n_2 + \cdots + n_k = n$, is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

2.5 Sterling's Formula

Theorem (Sterling's Formula)

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n}$$

(2.35, P.80) Approximate the following value for large n

$$\frac{2^n (n!)^2}{(2n)!} \sim \frac{\sqrt{\pi n}}{2^n}$$

(Exercise 1, P.80) $\binom{2n}{n} \frac{1}{2^{2n}} \approx \frac{1}{\sqrt{\pi n}}$

Solution for Review Problems 1,3,5,11.

1. Albert goes to the grocery store to buy fruit. There are seven different varieties of fruit and Albert is determined to buy no more than one of any variety. How many different orders can he place? Ans: 127
3. Virginia has one 1-dollar bill, one 2-dollar bill, one 5-dollar bill, one 10-dollar bill, and one 20-dollar bill. She decides to give some money to her son Brian without asking for change. How many choices does she have? Ans: 31
5. If four fair dice are tossed, what is the probability that they will show different faces? Ans: 360/1296
11. A list of all permutations of digits in {1,3,5,7,9} is put in increasing order. What is the 100^{th} number in the list? Ans: 91573

Solution for Review Problem 39 on P.85.

39. From the set of integers {1, 2, …, 100000} a number is selected at random.

What is the probability that the sum of its digits is 8?

Solution: Since the sum of the digits of 100,000 is 1, we ignore 100,000 and assume that all of the numbers have five digits by placing 0's in front of those with less than five digits. The following process establishes a one-to-one correspondence between such numbers, $d_1, d_2, d_3, d_4, d_5, \sum_{i=1}^{5} d_i = 8$, and placement of 8 identical objects into 5 distinguishable cells: Put d_1 of the objects into the first cell, d_2 of the objects into the second cell, d_3 into the third cell, and soon. Since this can be done in

$\binom{8+5-1}{5-1}=\binom{12}{4}$= 495 ways, the number of integers from the set {1,2,3,...,100000} in which the sum of the digits is 8 is 495. Hence the desired probability is 495/100,000= 0.00495.

Birthday Problem and Vandermonde Identity

- There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (excluding February 29), and that people's birthdays are independent (i.e., no twins in the room). What is the probability that two or more people in the room have the same birthday?
- Answer: 1- $\frac{365 \cdot 364 \cdots (365-k+1)}{(365)^k} \approx 0.99 \; when \; k = 57.$
- Vandermonde's Identity
- $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}, \qquad k \le \min\{m, n\}$

Story Proof Problems

- The team captain: $n\binom{n-1}{k-1} = k\binom{n}{k}, \quad k \leq n$.
- Bose-Einstein: The number of nonnegative integer solutions for

$$x_1 + x_2 + \cdots + x_n = k \quad is \quad \binom{n+k-1}{k}$$

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

$$\sum_{x=k}^{n} \binom{x}{k} = \binom{n+1}{k+1}, \; n \geq k$$

Story Proof Problems

P.33-34, CRC Press, Taylor & Francis Group (2015)

Introduction to Probability

Joseph K. Blitzstein (Harvard University)

Jessica Hwang (Stanford University)

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k} = \binom{2n}{n}$$

- Two groups of n boys and n girls. How many ways of selecting n persons from these $2n$ $persons$ to form a team?

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$$

- Consider choosing a committee of size n from two groups of size n each, where only one of the two groups has people eligible to become president.
- $\sum_{k=1}^{n} k\binom{n}{k}\binom{n}{n-k} = \cdots\cdots = n\binom{2n-1}{n-1}$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

- Imagine arranging a group of people by age, and then think about the oldest person in a chosen subgroup.
- $\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

(1) Proof by induction

(2a) $1 + 2 + \cdots + n = \binom{n+1}{2}$

(2b) $1^3 + 2^3 + \cdots + n^3 = 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2}$

(for 2a) Consider a round-robin tournament.

(for 2b) Imagine choosing a number between 1 and n and then choosing 3 numbers between 0 and n smaller than the original number, with replacement. Then consider cases based on how many distinct numbers were chosen.