1. Axioms of Probability

- Introduction: Uncertainty is the requirement of Probability Theory. We study the experiments for which the outcome cannot be predicted with certainty.

(a) National Tsing Hua university is located in Hsinchu,

(b) Tsinghua university is located in Beijing.

- Sample Space and Events
- Axioms of Probability
- Basic Theorems
- Continuity of Probability Function
- Probabilities 0 and 1
- Random Selection of Points from Intervals
- What is Simulation?

1.2 Sample Space and Events

- Sample Space (S): If the outcome of an experiment is not certain but all of its possible outcomes are predictable in advance, then the set of all these possible outcomes is called the *sample space*. A subset of the sample space is called an event.
- Example 1. Flipping a coin once, it is either a head or a tail. S={H (head), T (tail)}.
- Example 2. Throwing a die, there are six possible face values S={1, 2 ,3 ,4, 5, 6}.
- Example 3. Consider measuring the lifetime (in hours) of a light bulb.

  The sample space is T= $\{x: x \geq 0\}$.
- Example 4. Consider the outcome of tossing two dice. S=$\{(x, y): 1 \leq x,\ y \leq 6\}$.
- Example 5. The sum of outcomes by casting two dice. S=$\{r:\ 2 \leq r \leq 12\}$.
- Example 6. The grade of Linear Algebra course work. S={A, B, C, D, F}.
- Example 7. A light bulb survives at least 1000 hours. E=$\{x \in T: x \geq 1000\} \subseteq T$.

1.3 Axioms of Probability

- *Probability Axioms: Let S be the sample space of a random phenomenon. Suppose that to each event A of S, a number denoted by P(A) is associated with A. If P satisfies the following axioms, then it is called a probability and the number P(A) is said to be the probability of A.*

Axiom 1. $P(A) \geq 0$.

Axiom 2. P(S)=1.

Axiom 3. If $\{A_1, A_2, A_3, \cdots\}$ is a sequence of mutually exclusive events, i.e., $A_i \cap A_j = \emptyset$ *when* $i \neq j$, then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

- Theorems 1,2. $P(\emptyset) = 0$, and $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

1.3 Axioms of Probability

- Theorem 3. Let S be the sample space of an experiment. If S has *N* points that are equally likely to occur, then for an event A of S,

$$P(A)=\frac{N(A)}{N}$$

where N(A) is the number of points of A. (see the proof on P.16)

- Example 1. Let S the sample space of flipping a fair coin three times and A be the event of at least two heads appear. Then

S={HHH, HTH, HHT, HTT, THH, THT, TTH, TTT}, and P(A)=1/2.

Example 2. A number is selected at random from the set of {1, 2, 3, …, *N*}. What is the probability of the number is divisible by k, where $1 \leq k \leq N$?

$$\left(\text{Answer:} \frac{\text{floor}\left(\frac{N}{k}\right)}{N}\right).$$

1.4 Basic Theorems

- Theorem 4. Let S be the sample space of an experiment, A is an event, and A^c=S-A (the complement). then

  P(A^c)=1-P(A).

- Theorem 5. P($A \cup B$)=P(A)+P(B)-P($A \cap B$) and
- $$P(A) = P(A \cap B) + P(A \cap B^c).$$
- Example 1. Suppose that in a community of 400 adults, 300 bike or swim or do both, 160 swim, and 120 swim and bike, what is the probability, that an adult, selected at random from this community, bikes? (Answer: 260/400)
- Example 2. In a community, 32% of the population are male smokers; 27% are female smokers. What percentage of the population of this community smoke? (Answer: 59%)

1.4 Basic Theorems

- A08 on P.22: The admission office of a college admits only applicants whose high school GPA is at least 3.0 or whose SAT score 1200 or higher. If 38% of the applicants of this college have at least 3.0 GPA, 30% have a SAT of 1200 or higher, and 15% have both, what percentage of all the applicants are admitted to the college?

- A17 on P.23: Let S={a,b,c} be the sample space of an experiment. If P({a,b})=0.5, P({a,c})=0.7, find P({a}), P({b}), and P({c}), respectively.

- A35 on P.25: A number is selected randomly from the set {1,2,3,...,1000}. What is the probability that (a) it is divisible by 3 but not by 5; (b) it is divisible neither by 3 nor by 5?
- Q1 on P.27 for your practice.

1.5 Continuity of Probability Functions

- Definition of a Continuous Function: $f: R \to R$ if $lim_{x\to c}\mathrm{f}(x) = f(c)$
- An alternative description: $lim_{n\to\infty}\mathrm{f}(x_n) = f(lim_{n\to\infty}x_n)$

  for every convergent sequence $\{x_n\}_{n=1}^{\infty}$ in R.

□ An increasing sequence of events $E_1 \subseteq E_2 \subseteq \cdots E_n \subseteq E_{n+1} \subseteq \cdots$

□ A decreasing sequence of events $F_1 \supseteq F_2 \supseteq \cdots F_n \supseteq F_{n+1} \supseteq \cdots$

$\lim\limits_{n\to\infty} E_n = \bigcup_{i=1}^{\infty} E_i$ for an increasing sequence

$\lim\limits_{n\to\infty} F_n = \bigcap_{i=1}^{\infty} F_i$ for a decreasing sequence

1.5 Continuity of Probability Functions

Theorem 1.8 (Continuity of Probability Function)

For an increasing or decreasing sequence of events $\{E_n, n \geq 1\}$,

$\lim_{n\to\infty} P(E_n)$=P($\lim_{n\to\infty} E_n$)

Example 1.21 on P.29.

1.6 Probabilities 0 and 1

- If E and F are events with probabilities 1 and 0, respectively, it is not correct to say that E is the sample space and F is the empty space. Consider picking up a random point from the interval (0,1) such as
- $0.31415926\cdots \quad \mathrm{or}\ \frac{1}{3}$

  P(1/3 is selected among (0,1))=0 but {1/3}$\neq \emptyset$.

1.7 Random Selection of Points from Intervals

- A point is said to randomly selected from an interval (a, b) if any two subintervals of (a, b) that have the same length are equally likely to include the point. The probability associated with the event that the subinterval (s, t) contains the point is defined to be (t-s)/(b-a).
- Discussion of Example 1.22 (A Set that is Not an Event)

1.8 What is Simulation?

- Solving a scientific or an industrial problem usually involves mathematical analysis and/or simulation. To perform a simulation, we repeat an experiment a large number of times so assess the probability of an event or condition occurring. For example, to estimate the probability of at least one 6 appearing within four rolls of a die, we may do a large number of experiments rolling a die four times and calculate the number of times that at least one 6 is obtained.
- To simulate a problem that involves random phenomena, generating random numbers from the interval (0, 1) is essential. In almost every simulation of a probabilistic model, we will need to select random points from the interval (0, 1).

Selected Review Problems from P.39-43

1. The number of minutes it takes for a certain animal to react to a certain stimulus is a random number between 2 and 4.3. Find the probability that the reaction time of such an animal to this stimulus is no longer than 3.25 minutes.

Ans: $\frac{3.25-2}{4.3-2}=\frac{1.25}{2.3} \approx 0.54$

3. For a phone book, a phone number is selected at random. (a) What is event that the last digit is an odd number? (b) What is the event that the last digit is divisible by 3?

Ans: (a) A={1,3,5,7,9}; (b) B={0,3,6,9}.

5. A tutoring center, there are three computers that can be up or down at any given time. For $1 \leq i \leq 3$, let E_i be the event that computer i is up at a random time. In terms of E_i's, describe the event that at least two computers are up.

Ans: $E_1 E_2 E_3^c \cup E_1 E_2^c E_3 \cup E_1^c E_2 E_3 \cup E_1 E_2 E_3$

Selected Review Problems from P.39-43

13. In a midwest town, 80% of households have cable TV, 60% have an internet subscription, and 90% have at least one of these. What percentage of the households of this town have both cable TV and an internet subscription?

Ans: P(AB)=P(A)+P(B)-P($A \cup B$) =80%+60%-90%=0.5

15. The number of patients now in a hospital is 63. Of these 37 are male and 20 are for surgery. If among those who are for surgery 12 are male, how many of the 63 patients are neither male nor for surgery?

Ans: T=$M \cup F$, N(T)=63, N(M)=37, $\mathrm{N}(S \cap M) = 12, \mathrm{N}(S \cap F) = 8,$

N($S^c \cap F) = N(F) - N(S^c) = (63 - 37) - 8 = 18.$

Selected Review Problems from P.39-43

31. A number is selected at random from the set {1, 2, …, 150}. What is the probability that it is relatively prime to 150?

Ans: 1-P($A \cup B \cup C$)=1-[$\frac{75}{150}+\frac{50}{150}+\frac{30}{150}-\frac{25}{150}-\frac{15}{150}-\frac{10}{150}+\frac{5}{150}] = \frac{4}{15}$

35. The coefficients of the quadratic equation $ax^2+bx+c=0$ are determined by tossing a fair die three times (the first outcome is a, the second one is b, the third one is c). Find (a) the probability that the equation has no real roots? (b) the probability that the equation has two distinct real roots?

Ans: $\frac{173}{216}$ by computing $\{(a,b,c)\ in\ \{1,2,3,4,5,6\}\ \}\ such\ that\ b^2-4ac<0.$

Selected Review Problems from P.39-43

T5. For an experiment, $E \ and \ F$ are two events with $P(E) = 0.4$ and $P(E^c F^c) = 0.35$. Calculate $P(E^c \mathrm{F})$.

Ans: $P(E^c F) = P(F) - P(EF) = P(F) - [P(E) + P(F) - P(E \cup F)]$

$= -P(E) + P(E \cup F) = -P(E) + (1 - P(E^c F^c))$=-0.4+1-0.35=0.25.

Or by Wenn-Diagram.

T7. For an experiment with sample space $S = (0,2), for \ n \geq 1,$ let E_n=$\left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ and $P(E_n) = \frac{\mathrm{2n+1}}{\mathrm{3n}}$. For this experiment, find the probability that the event {1} occurs.

Ans: $P(E_n) = \frac{2}{3} \ as \ n \to \infty$.