2. Image Transform

- Fourier Transform
- Discrete Fourier Transform
- Fast Fourier Transform (FFT) O($N^2 \log(N)$)
- Discrete Cosine (Sine) Transform (*JPEG*)
- Discrete Wavelet Transform (*JP2*)
  - ♢ Haar transform
  - ♢ Daubechies' Four transform
- Singular Value Decomposition (SVD)
  - ♢ Spectrum Decomposition
  - Hotelling (Karhunen-L*o*eve) Transform
  - ♢ Principal Component Analysis (PCA)

♣ Other Transforms (Gabor, 9/7 and 5/3 Wavelets)

Introduction to Fourier Transform

Let $f(x)$ be a continuous function of a real variable x. The Fourier transform of $f(x)$ is defined by

$$F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi jux}dx$$

Given $F(u)$, $f(x)$ can be obtained by using the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} F(u)e^{2\pi jux}du$$

In practical applications, $f(x)$ is a real function, however, its Fourier transform $F(u)$ is in general *complex* and can he written as

$$F(u) = R(u) + jI(u) = |F(u)|e^{-j\phi(u)}$$

where $R(u)$ and $I(u)$ are the real and imaginary component, respectively. $|F(u)|$ is called the Fourier spectrum of $f(x)$, and $\phi(u)$ is called the phase angle of $f(x)$. $|F(u)|^2$ is called the *Fourier power spectrum* or *spectral density) of* $f(x)$.

Example: *Let* $f(x) = A$ *for* $0 \leq x \leq X$, *and* $f(x) = 0$ *otherwise. Then*

$$F(u) = \frac{A\sin(\pi uX)}{\pi u}e^{-j\pi uX}$$

$$|F(u)| = \frac{A\sin(\pi uX)}{\pi u} = AX|\frac{\sin(\pi uX)}{\pi uX}|$$

2D Fourier Transform

The extension of Fourier transform from 1D to 2D is obvious. The 2D Fourier transform of $f(x, y)$ can be written as

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi(ux+vy)j} dxdy$$

The 2D inverse Fourier transform of $F(u, v)$ is then

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{2\pi(ux+vy)j} dudv$$

For a real function $f(x, y)$, The Fourier transform $F(u, v)$ of $f(x, y)$ is in general complex *which can be written as*

$$F(u, v) = R(u, v) + jI(u, v) = |F(u, v)| e^{-j\phi(u,v)}$$

where $R(u, v)$ and $I(u, v)$ are the real and imaginary component, respectively. $|F(u, v)|$ is called the Fourier spectrum of $f(x, y)$ and $\phi(u, v)$ is called the phase angle of $f(x, y)$, $|F(u, v)|^2$ is called the Fourier power spectrum *or* spectral density *of $f(x, y)$.*

Example: *Let $f(x, y) = A$ if $0 \leq x \leq X, \quad 0 \leq y \leq Y$, and $f(x, y) = 0$ otherwise. Then*

$$F(u, v) = \frac{A}{\pi uv} \sin(\pi uX) e^{-j\pi uX} \sin(\pi vY) e^{-j\pi vY}$$

$$|F(u, v)| = AXY \left|\frac{\sin(\pi uX)}{\pi uX}\right| \left|\frac{\sin(\pi vY)}{\pi vY}\right|$$

Discrete Fourier Transform (DFT)

For many applications, Discrete Fourier transform (DFT) may be more useful than the continuous FT. The DFT of $f(x)$, $0 \le x \le N-1$ is defined as

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-2\pi uxj/N} \quad for \quad u = 0, 1, \cdots, N-1.$$

The inverse DFT of $F(u)$, $0 \le u \le N-1$ is defined as

$$f(x) = \sum_{u=0}^{N-1} F(u)e^{2\pi uxj/N} \quad for \quad x = 0, 1, \cdots, N-1.$$

For 2D DFT, we have

$$F(u,v) = \frac{1}{NM}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-2\pi[\frac{ux}{M}+\frac{vy}{N}]j} \quad for \quad 0 \le u \le M-1,\ 0 \le v \le N-1$$

The inverse DFT of $F(u,v)$, $0 \le u \le M-1$, $0 \le v \le N-1$, is defined as

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{2\pi[\frac{ux}{M}+\frac{vy}{N}]j} \quad for \quad 0 \le x \le M-1,\ 0 \le v \le N-1$$

Example: *Let $f(0)$=0.5, $f(1)$=0.75, $f(2)$=1.00, $f(3)$=1.25. Then*

$$F(0) = 0.875,\ F(1) = -0.125 + 0.125j,\ F(2) = -0.125,\ F(3) = -0.125 + 0.125j.$$

Introduction to Fast Fourier Transform

Assume that $N = 2^n$ *and* $M = \frac{N}{2}$. *The DFT is defined as*

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\omega_N^{ux}, \quad \textit{where} \;\; \omega_N = e^{-2\pi j/N}, \;\; \forall\; 0 \le u \le N$$

Then

$$F(u) = \frac{1}{2M}\sum_{x=0}^{2M-1} f(x)\omega_{2M}^{ux} = \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\omega_{2M}^{u(2x)} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\omega_{2M}^{u(2x+1)}\right]$$

Note that

$$\omega_{2M}^{2ux} = \omega_M^{ux}, \quad \omega_M^{u+M} = \omega_M^u, \quad \textit{and} \;\; \omega_{2M}^{u+M} = -\omega_{2M}^u$$

Define

$$F_{even}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)\omega_M^{ux}, \quad F_{odd}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\omega_M^{ux}$$

Thus

$$F(u) = \frac{1}{2}\left[F_{even}(u) + F_{odd}(u) * \omega_{2M}^u\right]$$

$$F(u+M) = \frac{1}{2}\left[F_{even}(u) - F_{odd}(u) * \omega_{2M}^u\right]$$

Computational Complexity of Fast Fourier Transform

To implement the above FFT, let

$$m(n) = \# \; of \; complex \; multiplications$$

$$a(n) = \# \; of \; complex \; additions/subtractions$$

Then

$$m(n) = 2m(n-1) + 2^{n-1}, \quad where \;\; m(0) = 0, \quad m(1) = 1.$$

$$a(n) = 2a(n-1) + 2^{n}, \quad where \;\; a(0) = 0, \quad a(1) = 2.$$

Then

$$\begin{aligned} m(n) &= 2m(n-1) + 2^{n-1} \\ &= 2[2m(n-2) + 2^{n-2}] + 2^{n-1} \\ &= \ldots \\ &= 2^{n-1} \times m(1) + (n-1) \times 2^{n-1} \\ &= 2^{n-1} \times n \\ &= \tfrac{1}{2} N \log_2 N \end{aligned}$$

Similarly

$$a(n) = 2^{n} \log_2 N = N \log_2 N$$

Implement Inverse FFT by Forward FFT Algorithm

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi uxj/N} \quad for \quad 0 \le u \le N-1$$

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{2\pi uxj/N} \quad for \quad 0 \le x \le N-1$$

$$f^*(x) = \sum_{u=0}^{N-1} F^*(u) e^{-2\pi uxj/N} \quad for \quad 0 \le x \le N-1$$

Then

$$\frac{1}{N} f^*(x) = \frac{1}{N} \sum_{u=0}^{N-1} F^*(u) e^{-2\pi uxj/N}$$

Similarly

$$\frac{1}{N^2} f^*(x,y) = \frac{1}{N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F^*(u,v) e^{-2\pi(ux+vy)j/N}$$

Discrete Cosine Transform (DCT)

The 1D DCT of $f(x)$, $0 \le x \le n-1$ *can be defined as*

$$C(0) = \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} f(x)$$

$$C(u) = \frac{\sqrt{2}}{\sqrt{n}} \sum_{x=0}^{n-1} f(x) \cos\left[\frac{(2x+1)u\pi}{2n}\right], \quad u = 1, 2, \cdots, n-1$$

The inverse DCT of $C(u)$, $0 \le u \le n-1$, *is then defined as*

$$f(x) = \frac{C(0)}{\sqrt{n}} + \frac{\sqrt{2}}{\sqrt{n}} \sum_{u=1}^{n-1} C(u) \cos\left[\frac{(2x+1)u\pi}{2n}\right], \quad x = 0, 1, \cdots, n-1$$

The n-point DCT Implementation could be done by applying 2n-point FFT.

$$C(0) = \frac{1}{\sqrt{n}} \sum_{x=0}^{n-1} f(x)$$

$$C(u) = \frac{\sqrt{2}}{\sqrt{n}} \times \mathit{Real}\left[e^{-\pi u j/2n} \cdot \sum_{x=0}^{2n-1} f(x) e^{-\pi u x j/n}\right]$$

where $u = 1, 2, \cdots, n-1, \; f(x) = 0 \;\; \forall \; x = n, n+1, \cdots, 2n-1$

2D Discrete Cosine Transform

The corresponding 2D DCT pair is defined as

$$C(u,v) = \alpha_u \alpha_v \sum_{x=0}^{n-1}\sum_{y=0}^{n-1} f(x,y)\cos[\frac{(2x+1)u\pi}{2n}]\cos[\frac{(2y+1)v\pi}{2n}] \quad for \quad 0 \le u,v \le n-1$$

The inverse DCT of $C(u,v)$ is defined as

$$f(x,y) = \sum_{u=0}^{n-1}\sum_{v=0}^{n-1} \alpha_u \alpha_v C(u,v)\cos[\frac{(2x+1)u\pi}{2n}]\cos[\frac{(2y+1)v\pi}{2n}] \quad for \quad 0 \le x,y \le n-1$$

where

$$\alpha_u = \begin{cases} \frac{1}{\sqrt{n}} & if \;\; u = 0, \\ \\ \frac{\sqrt{2}}{\sqrt{n}} & if \;\; 1 \le u \le n-1 \end{cases}$$

In practice, $n = 8$ is used, we have

$$C(u,v) = \frac{\gamma_u \gamma_v}{4}\sum_{x=0}^{7}\sum_{y=0}^{7} f(x,y)\cos[\frac{(2x+1)u\pi}{16}]\cos[\frac{(2y+1)v\pi}{16}]$$

$$f(x,y) = \frac{1}{4}\sum_{u=0}^{7}\sum_{v=0}^{7} \gamma_u \gamma_v C(u,v)\cos[\frac{(2x+1)u\pi}{16}]\cos[\frac{(2y+1)v\pi}{16}]$$

where $\gamma_0 = \frac{1}{\sqrt{1}}$, and $\gamma_v = 1$ if $v = 1, 2, \cdots, n-1$

DCT in Matrix Forms

$Y = CXC^t$, where X is an 8×8 image block and $C = Q_8$ and Q_4 are orthogonal matrices defined as follows.

$$Q_8 = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ cos\frac{\pi}{16} & cos\frac{3\pi}{16} & cos\frac{5\pi}{16} & cos\frac{7\pi}{16} & cos\frac{9\pi}{16} & cos\frac{11\pi}{16} & cos\frac{13\pi}{16} & cos\frac{15\pi}{16} \\ cos\frac{2\pi}{16} & cos\frac{6\pi}{16} & cos\frac{10\pi}{16} & cos\frac{14\pi}{16} & cos\frac{18\pi}{16} & cos\frac{22\pi}{16} & cos\frac{26\pi}{16} & cos\frac{30\pi}{16} \\ cos\frac{3\pi}{16} & cos\frac{9\pi}{16} & cos\frac{15\pi}{16} & cos\frac{21\pi}{16} & cos\frac{27\pi}{16} & cos\frac{33\pi}{16} & cos\frac{39\pi}{16} & cos\frac{45\pi}{16} \\ cos\frac{4\pi}{16} & cos\frac{12\pi}{16} & cos\frac{20\pi}{16} & cos\frac{28\pi}{16} & cos\frac{36\pi}{16} & cos\frac{44\pi}{16} & cos\frac{52\pi}{16} & cos\frac{60\pi}{16} \\ cos\frac{5\pi}{16} & cos\frac{15\pi}{16} & cos\frac{25\pi}{16} & cos\frac{35\pi}{16} & cos\frac{45\pi}{16} & cos\frac{55\pi}{16} & cos\frac{65\pi}{16} & cos\frac{75\pi}{16} \\ cos\frac{6\pi}{16} & cos\frac{18\pi}{16} & cos\frac{30\pi}{16} & cos\frac{42\pi}{16} & cos\frac{54\pi}{16} & cos\frac{66\pi}{16} & cos\frac{78\pi}{16} & cos\frac{90\pi}{16} \\ cos\frac{7\pi}{16} & cos\frac{21\pi}{16} & cos\frac{35\pi}{16} & cos\frac{49\pi}{16} & cos\frac{63\pi}{16} & cos\frac{77\pi}{16} & cos\frac{91\pi}{16} & cos\frac{105\pi}{16} \end{bmatrix}$$

$$Q_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ cos\frac{\pi}{8} & cos\frac{3\pi}{8} & cos\frac{5\pi}{8} & cos\frac{7\pi}{8} \\ cos\frac{2\pi}{8} & cos\frac{6\pi}{8} & cos\frac{10\pi}{8} & cos\frac{14\pi}{8} \\ cos\frac{3\pi}{8} & cos\frac{9\pi}{8} & cos\frac{15\pi}{8} & cos\frac{21\pi}{8} \end{bmatrix} \approx \frac{1}{26}\begin{bmatrix} 13 & 13 & 13 & 13 \\ 17 & 7 & -7 & -17 \\ 13 & -13 & -13 & 13 \\ 7 & -17 & 17 & -7 \end{bmatrix}$$

Note that in Matlab, *the matrix $Q_8 = C$ can be acquired by $C = dctmtx(8)$.*

Quantization Table and An 8×8 Image Block

16	11	10	16	24	40	51	61
12	12	14	19	26	58	60	55
14	13	16	24	40	57	69	56
14	17	22	29	51	87	80	62
18	22	37	56	68	109	103	77
24	35	55	64	81	104	113	92
49	64	78	87	103	121	120	101
72	92	95	98	112	100	103	99

Table 1: **Quantization Table for DCT Coefficients**

139	144	149	153	155	155	155	155
144	151	153	156	159	156	156	156
150	155	160	163	158	156	156	156
159	161	162	160	160	159	159	159
159	160	161	162	162	155	155	155
161	161	161	161	160	157	157	157
162	162	161	163	162	157	157	157
162	162	161	161	163	158	158	158

Table 2: **An 8×8 Image Block X**

An 8×8 Image Block and Its DCT Coefficients

139	*144*	*149*	*153*	*155*	*155*	*155*	*155*
144	*151*	*153*	*156*	*159*	*156*	*156*	*156*
150	*155*	*160*	*163*	*158*	*156*	*156*	*156*
159	*161*	*162*	*160*	*160*	*159*	*159*	*159*
159	*160*	*161*	*162*	*162*	*155*	*155*	*155*
161	*161*	*161*	*161*	*160*	*157*	*157*	*157*
162	*162*	*161*	*163*	*162*	*157*	*157*	*157*
162	*162*	*161*	*161*	*163*	*158*	*158*	*158*

Table 3: **An 8×8 Image Block X**

235.6	*-1.0*	*-12.1*	*-5.2*	*2.1*	*-1.7*	*-2.7*	*1.3*
-22.6	*-17.5*	*-6.2*	*-3.2*	*-2.9*	*-0.1*	*0.4*	*-1.2*
-10.9	*-9.3*	*-1.6*	*1.5*	*0.2*	*-0.9*	*-0.6*	*-0.1*
-7.1	*-1.9*	*0.2*	*1.5*	*0.9*	*-0.1*	*-0.0*	*0.3*
-0.6	*-0.8*	*1.5*	*1.6*	*-0.1*	*-0.7*	*0.6*	*1.3*
1.8	*-0.2*	*1.6*	*-0.3*	*-0.8*	*1.5*	*1.0*	*-1.0*
-1.3	*-0.4*	*-0.3*	*-1.5*	*-0.5*	*1.7*	*1.1*	*-0.8*
-2.6	*1.6*	*-3.8*	*-1.8*	*1.9*	*1.2*	*-0.6*	*-0.4*

Table 4: **DCT Coefficients** $Q_8 X Q_8^t$ **of Image Block X**

Representation of Quantized DCT Coefficients

235.6	*-1.0*	*-12.1*	*-5.2*	*2.1*	*-1.7*	*-2.7*	*1.3*
-22.6	*-17.5*	*-6.2*	*-3.2*	*-2.9*	*-0.1*	*0.4*	*-1.2*
-10.9	*-9.3*	*-1.6*	*1.5*	*0.2*	*-0.9*	*-0.6*	*-0.1*
-7.1	*-1.9*	*0.2*	*1.5*	*0.9*	*-0.1*	*-0.0*	*0.3*
-0.6	*-0.8*	*1.5*	*1.6*	*-0.1*	*-0.7*	*0.6*	*1.3*
1.8	*-0.2*	*1.6*	*-0.3*	*-0.8*	*1.5*	*1.0*	*-1.0*
-1.3	*-0.4*	*-0.3*	*-1.5*	*-0.5*	*1.7*	*1.1*	*-0.8*
-2.6	*1.6*	*-3.8*	*-1.8*	*1.9*	*1.2*	*-0.6*	*-0.4*

Table 5: **DCT Coefficients** $Q_8 X Q_8^t$ **of Image Block X**

15	*0*	*-1*	*0*	*0*	*0*	*0*	*0*
-2	*-1*	*0*	*0*	*0*	*0*	*0*	*0*
-1	*-1*	*0*	*0*	*0*	*0*	*0*	*0*
-1	*0*	*0*	*0*	*0*	*0*	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*

Table 6: **Quantized DCT Coefficients for Image Block X**

Suppose that the quantized DC *coefficient for the last block is* 12.

(**S,2,3**), *(1,2,-2), (0,1,-1), (0,1,-1), (0,1,-1), (2,1,-1), (0,1,-1) EOB*

DAUB4 Wavelet Transformation Matrix

$$c_0 = \tfrac{1+\sqrt{3}}{4\sqrt{2}},\ \ c_1 = \tfrac{3+\sqrt{3}}{4\sqrt{2}},\ \ c_2 = \tfrac{3-\sqrt{3}}{4\sqrt{2}},\ \ c_3 = \tfrac{1-\sqrt{3}}{4\sqrt{2}}$$

$$W = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & & & & & & & & \\ c_3 & -c_2 & c_1 & -c_0 & & & & & & & & \\ & & c_0 & c_1 & c_2 & c_3 & & & & & & \\ & & c_3 & -c_2 & c_1 & -c_0 & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & \ddots & & & & \\ & & & & & & & & c_0 & c_1 & c_2 & c_3 \\ & & & & & & & & c_3 & -c_2 & c_1 & -c_0 \\ c_2 & c_3 & & & & & & & & & c_0 & c_1 \\ c_1 & -c_0 & & & & & & & & & c_3 & -c_2 \end{bmatrix}$$

$$W = DAUB4, \quad {c_0}^2 \ + \ {c_1}^2 \ + \ {c_2}^2 \ + \ {c_3}^2 \ = \ 1.$$

$$Y \ \leftarrow \ P \ *_4 \ W \ *_3 \ X \ *_1 \ W^t \ *_2 \ Q$$

A 3-scale Daubechies' four transform of an 8×8 image

64	90	83	88	127	174	174	179
67	93	89	82	125	175	180	178
65	79	79	69	127	169	180	181
59	74	91	62	117	170	182	181
57	82	98	57	102	162	180	180
59	75	104	58	90	163	187	183
59	79	105	67	74	159	189	182
61	76	100	67	65	131	188	186

935	-316	61	-84	2	-11	10	-73
21	-3	0	-75	24	-9	15	-71
-4	5	11	3	46	-39	25	-76
29	7	18	0	36	-52	37	-74
-2	-4	0	2	-6	7	-1	3
4	-2	-4	-5	2	1	-2	-4
2	2	10	-1	1	-10	0	-1
-1	7	18	-6	-14	12	-11	0

Haar Wavelet Transformation Matrix

$$c_0 = 1/\sqrt{2},\ c_1 = 1/\sqrt{2}.$$

$$H = \begin{bmatrix} c_0 & c_1 & & & & & & & & \\ c_0 & -c_1 & & & & & & & & \\ & & c_0 & c_1 & & & & & & \\ & & c_0 & -c_1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & c_0 & c_1 & & \\ & & & & & & c_0 & -c_1 & & \\ & & & & & & & & c_0 & c_1 \\ & & & & & & & & c_0 & -c_1 \end{bmatrix}$$

$$Y \leftarrow P \ *_4 \ H \ *_3 \ X \ *_1 \ H^t \ *_2 \ Q$$

Example of 3-scale Haar transform of an 8×8 image

64	90	83	88	127	174	174	179
67	93	89	82	125	175	180	178
65	79	79	69	127	169	180	181
59	74	91	62	117	170	182	181
57	82	98	57	102	162	180	180
59	75	104	58	90	163	187	183
59	79	105	67	74	159	189	182
61	76	100	67	65	131	188	186

926	-325	-12	-62	-26	0	-48	-1
26	-19	-26	-131	-14	19	-47	0
19	1	0	7	-20	43	-66	2
-4	18	4	24	-17	35	-75	4
-2	0	0	-2	0	-5	0	-2
4	-2	4	0	0	-8	4	0
2	-2	5	-4	-3	-2	5	-1
0	2	17	-1	-2	2	-9	1

Singular Value Decomposition (SVD) + PCP

Spectrum Decomposition Theorem: $A = W\Lambda W^t$

Every symmetric matrix can be diagonalized.

SVD Theorem: $A = U\Sigma V^t = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^t$

Each matrix $A \in R^{m\times n}$ can be decomposed as $A = U\Sigma V^t$, where both $U \in R^{m\times m}$ and $V \in R^{n\times n}$ are orthogonal. Moreover, $\Sigma \in R^{m\times n} = diag[\sigma_1, \sigma_2, \ldots, \sigma_k, 0, \ldots, 0]$ is essentially diagonal with the singular values satisfying $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > 0$, and $\sigma_j = 0 \;\forall\; j \geq (k+1)$.

Example:

$$A = \begin{bmatrix} -2 & -2 & 6 & 6 \\ -2 & 2 & -6 & -6 \\ 6 & -6 & 5 & -1 \\ 6 & -6 & -1 & 5 \end{bmatrix}$$

Then $\Sigma(A) = diag[16, 8, 6, 0]$, $\lambda(A) = \{16, -8, 6, 0\}$.

$$U = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Gabor Filter(s) and Transform(s)

A 2-dimensional (symmetrical) gabor filter can be defined as

$$f(x,y) = \exp\left(-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]\right) \times \cos[2\pi\mu_0(x\cos\theta + y\sin\theta)] \tag{1}$$

where μ_0 *is the frequency of a sinusoidal plane and* σ_x *and* σ_y *are corresponding to the variance of the Gaussian distribution (shape). Simple computations lead to the Fourier transform which possesses a similar form as the gabor filter has.*

$$\begin{aligned} F(u,v) &= \pi\sigma_x\sigma_y \left\{\exp\left(-\tfrac{1}{2}\left[\tfrac{(u-\mu_0\cos\theta)^2}{\sigma_u^2} + \tfrac{(v-\mu_0\sin\theta)^2}{\sigma_v^2}\right]\right)\right. \\ &+ \exp\left(-\tfrac{1}{2}\left[\tfrac{(u+\mu_0\cos\theta)^2}{\sigma_u^2} + \tfrac{(v+\mu_0\sin\theta)^2}{\sigma_v^2}\right]\right) \end{aligned}$$

where $\sigma_u = \frac{1}{2\pi\sigma_x}$, $\sigma_v = \frac{1}{2\pi\sigma_y}$.

$$I[x,y] * F_j[x,y] = InvFT\{FT(I[x,y]) \times FT(F_j[x,y])\}$$