Error Analysis of Lagrange Polynomial Interpolation

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}\prod_{k=0}^{n}(x - x_k),$$

where $P(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$.

Proof Note first that if $x = x_k$ for $k = 0, 1, \ldots, n$, then $f(x_k) = P(x_k)$ and choosing $\xi(x_k)$ arbitrarily in (a, b) yields Eq. (3.4). If $x \neq x_k$ for any $k = 0, 1, \ldots, n$, define the function g for t in $[a, b]$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t - x_0)(t - x_1)\cdots(t - x_n)}{(x - x_0)(x - x_1)\cdots(x - x_n)}$$

$$= f(t) - P(t) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}.$$

Since $f \in C^{n+1}[a, b]$, $P \in C^{\infty}[a, b]$, and $x \neq x_k$ for any k, it follows that $g \in C^{n+1}[a, b]$. For $t = x_k$,

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x_k - x_i)}{(x - x_i)}$$

$$= 0 - [f(x) - P(x)]\cdot 0 = 0.$$

Moreover,

$$g(x) = f(x) - P(x) - [f(x) - P(x)]\prod_{i=0}^{n}\frac{(x - x_i)}{(x - x_i)}$$

$$= f(x) - P(x) - [f(x) - P(x)] = 0.$$

Thus, $g \in C^{n+1}[a, b]$ and g vanishes at the $n + 2$ distinct numbers $x, x_0, x_1, \ldots, x_n$. By the Generalized Rolle's theorem, there exists ξ in (a, b) for which $g^{(n+1)}(\xi) = 0$. So,

$$(3.5)\quad 0 = g^{(n+1)}(\xi) = f^{(n+1)}(\xi) - P^{(n+1)}(\xi) - [f(x) - P(x)]\frac{d^{n+1}}{dt^{n+1}}\left[\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)}\right]_{t=\xi}.$$

Since P is a polynomial of degree at most n, the $(n + 1)$st derivative, $P^{(n+1)}$, is identically zero. Also, $\prod_{i=0}^{n}[(t - x_i)/(x - x_i)]$ is a polynomial of degree $(n + 1)$, so

$$\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} = \left[\frac{1}{\prod_{i=0}^{n}(x - x_i)}\right]t^{n+1} + (\text{lower-degree terms in } t),$$

and

$$\frac{d^{n+1}}{dt^{n+1}}\prod_{i=0}^{n}\frac{(t - x_i)}{(x - x_i)} = \frac{(n + 1)!}{\prod_{i=0}^{n}(x - x_i)}.$$

Equation (3.5) now becomes

$$0 = f^{(n+1)}(\xi) - 0 - [f(x) - P(x)]\frac{(n - 1)!}{\prod_{i=0}^{n}(x - x_i)}$$

and, upon solving for $f(x)$,

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi)}{(n + 1)!}\prod_{i=0}^{n}(x - x_i).$$ ▪ ▪ ▪

The error formula in Theorem 3.3 is an important theoretical result because Lagrange polynomials are used extensively for deriving numerical differentiation and integration methods. Error bounds for these techniques are obtained from the Lagrange error formula.

Note that the error form for the Lagrange polynomial is quite similar to that for the Taylor polynomial. The Taylor polynomial of degree n about x_0 concentrates all the known information at x_0 and has an error term of the form

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1},$$

The Lagrange polynomial of degree n uses information at the distinct numbers $x_0, x_1, \ldots, x_n$ and, in place of $(x-x_0)^n$, its error formula uses a product of the $n+1$ terms $(x-x_0), (x-x_1), \ldots, (x-x_n)$:

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n).$$

The specific use of this error formula is restricted to those functions whose derivatives have known bounds.

EXAMPLE 2 Suppose a table is to be prepared for the function $f(x) = e^x$, $0 \le x \le 1$. Assume the number of decimal places to be given per entry is $d \ge 6$ and that the difference between adjacent x-values, the step size, is h. What should h be for linear interpolation (i.e., the Lagrange polynomial of degree 1) to give an absolute error of at most 10^{-6}?

Let $x \in [0, 1]$ and suppose j satisfies $x_j \le x \le x_{j+1}$. From Eq. (3.4), the error in linear interpolation is

$$|f(x) - P(x)| = \left|\frac{f^{(2)}(\xi)}{2!}(x-x_j)(x-x_{j+1})\right| = \frac{|f^{(2)}(\xi)|}{2}\,|(x-x_j)|\,|(x-x_{j+1})|.$$

Since the step size is h, it follows that $x_j = jh$, $x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \le \frac{|f^{(2)}(\xi)|}{2!}|(x-jh)(x-(j+1)h)|.$$

Hence,

$$|f(x) - P(x)| \le \tfrac{1}{2}\max_{\xi\in[0,1]} e^{\xi} \max_{x_j\le x\le x_{j+1}} |(x-jh)(x-(j+1)h)|.$$

$$\le \tfrac{1}{2}e \max_{x_j\le x\le x_{j+1}} |(x-jh)(x-(j+1)h)|.$$

By considering $g(x) = (x-jh)(x-(j+1)h)$ for $jh \le x \le (j+1)h$ and using techniques of calculus (see Exercise 24),

$$\max_{x_j\le x\le x_{j+1}} |g(x)| = |g((j+\tfrac{1}{2})h)| = \frac{h^2}{4}.$$

Consequently, the error in linear interpolation is bounded by

$$|f(x) - P(x)| \le \frac{eh^2}{8}.$$

Hence it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \le 10^{-6},\ h^2 \le \frac{8}{e}\cdot 10^{-6}, \quad h^2 < 2.944\times 10^{-6}, \quad \text{or} \quad h < 1.72\times 10^{-3}.$$

Letting $h = 0.001$ would be one logical choice for the step size. ■ ■ ■