1. (25%) Prove that if $P = NP$, then $PATH$ is NP-complete.\footnote{An immediate corollary of this: If $PATH$ is not NP-complete, then $P \neq NP$.}

2. (25%) Let $LPATH$ denote the language:

$$LPATH = \{ (G, s, t, k) \mid G \text{ contains a simple path of length at least } k \text{ from } s \text{ to } t \}.$$ 

Show that $LPATH$ is NP-complete. (Hint: Reduction from $HAMPATH$.)

3. Let $\phi$ be a cnf-formula. An assignment to the variables of $\phi$ is called not-all-equal if in each clause, at least one literal is TRUE and at least one literal is FALSE.

Let $\neq SAT$ be the language:

$$\neq SAT = \{ \langle \phi \rangle \mid \phi \text{ is a cnf-formula which has a satisfying not-all-equal assignment} \}.$$ 

For example,

- $\phi_1 = (u \lor v) \land (v \lor x)$ is in $\neq SAT$;
- $\phi_2 = (u \lor v) \land (\neg u \lor v)$ is not in $\neq SAT$.

(25%) Show that $\neq SAT$ is NP-complete.

Hint: Reduction from $CNF$-$SAT$ by replacing each clause $C_i$

$$(x_1 \lor x_2 \lor \cdots \lor x_k)$$

with the two clauses

$$(x_1 \lor x_2 \lor \cdots \lor x_{k-1} \lor z_i) \text{ and } (\neg z_i \lor x_k)$$

4. (25%) Let $S$ be a finite set and $C = \{C_1, C_2, \ldots, C_k\}$ be a collection of subsets of $S$, for some $k > 0$. We say $S$ is two-colorable with respect to $C$ if we can color the elements of $S$ in either red or blue, such that each subset $C_i$ contains at least a red element and at least a blue element.

Let $2COLOR$ denote the language:

$$2COLOR = \{ \langle S, C \rangle \mid S \text{ is two-colorable with respect to } C \}.$$ 

Show that $2COLOR$ is NP-complete. (Reduction from which NP-complete problem??)

5. (Further Studies: No marks) If $P = NP$, will all languages in $P$ become NP-complete?

6. (Further Studies: No marks) Let $CNF_k = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable cnf-formula where each variable appears in at most } k \text{ places} \}$.

(a) Show that $CNF_2 \in P$.

(b) Show that $CNF_3$ is NP-complete.