1. **Ans.** Suppose on the contrary that $T$ is decidable. Let $R$ be its decider. Then, the following TM $Q$ is a decider for $A_{TM}$:

$$Q = \text{"On input } \langle M, w \rangle, \text{"}$$

1. Construct a TM $M'$ as follows:
   - $M' = \text{"On input } x, \text{"}$
   - 1. If $x \neq 011$, accept.
   - 2. Run $M$ on $w$.
   - 3. If $M$ accepts $w$, accept.
2. Run $R$ to decide if $\langle M' \rangle$ is in $T$.
3. If yes (i.e., $R$ accepts), accept.
4. Else, reject.

It is easy to check that $Q$ runs in finite steps. Also, in Step 1, $M'$ has the property that:

(i) If $M$ accepts $w$, $L(M') = \Sigma^*$, so that $\langle M' \rangle \in T$.

(ii) Else, $L(M') = \Sigma^* - \{011\}$, so that $\langle M' \rangle \notin T$.

So, if $Q$ accepts $\langle M, w \rangle$, it must mean that $R$ accepts $\langle M' \rangle$, which implies that $\langle M' \rangle \in T$, which implies $M$ accepts $w$. On the other hand, if $Q$ rejects $\langle M, w \rangle$, $R$ rejects $\langle M' \rangle$, which in turn implies that $M$ does not accept $w$.

Thus, $Q$ is a decider for $A_{TM}$, and a contradiction occurs. So, we conclude that $T$ is undecidable.

2. In the *silly Post Correspondence Problem*, we see that if a set of dominoes $S$ is in $SPCP$ if and only if $S$ contains a piece whose top string matches exactly the bottom string. Thus, we can easily design a TM that uses finite steps to check such a piece exists. So, $SPCP$ is decidable.

3. $(\Rightarrow)$ If $A \leq_m A_{TM}$, then $A$ is Turing-recognizable because $A_{TM}$ is Turing recognizable.

$(\Leftarrow)$ If $A$ is Turing-recognizable, then there exists some TM $R$ that recognizes $A$. That is, $R$ would receive an input $w$ and accept if $w$ is in $A$ (otherwise $R$ does not accept). To show that $A \leq_m A_{TM}$, we design a TM that does the following: On input $w$, writes $\langle R, w \rangle$ on the tape and halts. It is easy to check that $\langle R, w \rangle$ is in $A_{TM}$ if and only if $w$ is in $A$. Thus, we get a mapping reduction of $A$ to $A_{TM}$.

4. $(\Rightarrow)$ If $A \leq_m 0^*1^*$, then $A$ is decidable because $0^*1^*$ is a decidable language.

$(\Leftarrow)$ If $A$ is decidable, then there exists some TM $R$ that decides $A$. That is, $R$ would receive an input $w$ and accept if $w$ is in $A$, reject if $w$ is not in $A$. To show $A \leq_m 0^*1^*$, we design a TM $Q$ that does the following: On input $w$, runs $R$ on $w$. If $R$ accepts, outputs $01$; otherwise, outputs $10$. It is easy to check that:

$$w \in A \iff \text{output of } Q \in 0^*1^*.$$

Thus, we obtain a mapping reduction of $A$ to $0^*1^*$. 

1
5. Let $J = \{ w \mid \text{either } w = 0x \text{ for some } x \in A_{TM}, \text{ or } w = 1y \text{ for some } y \notin A_{TM} \}$.

- We first show that $A_{TM} \leq_m J$. To do so, we design the following TM $Q$: On input $\langle M, w \rangle$, write 0 followed by $\langle M, w \rangle$ in the tape and halts. It is easy to check that:

  $\langle M, w \rangle \in A_{TM} \iff \text{output of } Q \in J.$

  Thus, we obtain a mapping reduction of $A_{TM}$ to $J$.

- We next show that $A_{TM} \leq_m \overline{J}$. To do so, we design the following TM $R$: On input $\langle M, w \rangle$, write 1 followed by $\langle M, w \rangle$ in the tape and halts. It is easy to check that:

  $\langle M, w \rangle \in A_{TM} \iff \text{output of } R \in J.$

  Equivalently, we have:

  $\langle M, w \rangle \in A_{TM} \iff \text{output of } R \in \overline{J}.$

  Thus, we obtain a mapping reduction of $A_{TM}$ to $\overline{J}$.

- Since $A_{TM} \leq_m J$, we have $\overline{A_{TM}} \leq_m \overline{J}$. This shows that $\overline{J}$ is non-Turing-recognizable because $A_{TM}$ is non-Turing-recognizable.

  Similarly, since $A_{TM} \leq_m \overline{J}$, we have $\overline{A_{TM}} \leq_m J$. So, this shows that $J$ is non-Turing-recognizable.