CS5314
Randomized Algorithms

Lecture 20: Probabilistic Method
(Lovasz Local Lemma)
Objectives

• Introduce Lovasz Local Lemma (LLL)
  - one of the most elegant and useful tools in the probabilistic method

• Two versions:
  - symmetric case
  - general case
Let $E_1, E_2, \ldots, E_n$ be a set of $\text{BAD}$ events

- Suppose each occurs with prob $< 1$

**Fact:** If they are mutually independent, it is easy to see that

$$\Pr(\text{no BAD events}) > 0 \quad \text{[why?]}

- However, in many natural scenario, the $\text{BAD}$ events are not mutually independent

**Problem:** Can we still easily show that

$$\Pr(\text{no BAD events}) > 0 ?$$
Lovasz Local Lemma (2)

- In general, probably not...
- But, if there are not many dependency among the BAD events, then the set of events are ‘roughly’ mutually independent
  - we may still be able to show \( \Pr(\text{no BAD events}) > 0 \) ...

- Lovasz Local Lemma gives sufficient conditions when we can do so ...
  - It relies on a concept of dependency graph defined as follows (next slide)
Let \( E \) be an event

**Definition:** \( E \) is mutually independent of a set of events \( \{E_1, E_2, \ldots, E_n\} \) if for any \( I \subseteq [1,n] \), \( \Pr( E | \bigcap_{j \in I} E_j ) = \Pr(E) \)

**Definition:** A dependency graph for a set of events \( \{E_1, E_2, \ldots, E_n\} \) is a graph \( G=(V,E) \), \( V = \{1,2,\ldots,n\} \) such that for any \( j \), \( E_j \) is mutually independent of the events \( \{ E_k \ | \ (j,k) \notin E \} \)
Test your understanding:

1. Let $S$ be a set of pair-wise independent events. Is a graph with no edges always a dependency graph of $S$?

2. Let $S$ be a set of events. Is the dependency graph of $S$ unique?

The answers are **NO** for both questions...
Consider flipping a fair coin twice.
Let $E_1$ = the first flip is head
$E_2$ = the second flip is tail
$E_3$ = the two flips are the same

→ the events are pairwise independent

We see that if a graph has less than 2 edges, it must not be a dependency graph

On the other hand, any graph with 2 or more edges is a dependency graph !!!
Lovasz Local Lemma
(Symmetric Case)

Theorem: Let $G$ be a dependency graph of a set of BAD events $\{E_1, E_2, \ldots, E_n\}$. If

(i) $\Pr(E_j) \leq p < 1$ for each $E_j$,
(ii) $1 \leq \maxdeg(G) \leq d$, and
(iii) $4pd \leq 1$

then $\Pr(\text{no BAD events}) > 0$

Remark: If $\maxdeg(G) = 0$, then $\Pr(\text{no BAD events}) > 0$ since all events are mutually independent
Proof

Let $S = \{s_1, s_2, \ldots\}$ be a subset of $\{1, 2, \ldots, n\}$

• The proof is based on induction

• In particular, we show two statements are true alternately:

  (1) $\Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) \leq 2p$ for all $S$, with $|S| = 0, 1, 2, \ldots, n-1$

  (2) $\Pr(\bigcap_{j \in S} \neg E_j ) > 0$ for all $S$, with $|S| = 1, 2, \ldots, n$
Proof (2)

• The base case(s) are: 1\textsuperscript{st} statement with \(|S|=0\), and 2\textsuperscript{nd} statement with \(|S|=1\)

• For the inductive steps:

(1) Assume 1\textsuperscript{st} statement is true for \(|S| \leq h\) and 2\textsuperscript{nd} statement is true for \(|S| \leq h+1\)

\(\Rightarrow\) prove 1\textsuperscript{st} statement is true for \(|S|=h+1\)

(2) Assume 1\textsuperscript{st} statement is true for \(|S| \leq h+1\) and 2\textsuperscript{nd} statement is true for \(|S| \leq h+1\),

\(\Rightarrow\) prove 2\textsuperscript{nd} statement is true for \(|S|=h+2\)
Consequently, by induction,
we can prove the 1\textsuperscript{st} statement when $|S|=1$,
and then the 2\textsuperscript{nd} statement when $|S|=2$,
and then the 1\textsuperscript{st} statement when $|S|=2$,
and then the 2\textsuperscript{nd} statement when $|S|=3$,
and so on...
Proof: Base Cases

Base Case 1: 1st statement, \(|S| = 0\)

In this case, we have

\[ \Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) = \Pr( E_k ) \leq p \leq 2p \]

⇒ So this case is true

Base Case 2: 2nd statement, \(|S| = 1\)

In this case, we have

\[ \Pr( \bigcap_{j \in S} \neg E_j ) = 1 - \Pr( E_{s_1} ) \geq 1 - p > 0 \]

⇒ So this case is true
Proof: Inductive Case 1

Inductive Case 1: Assume 1\textsuperscript{st} statement is true for $|S| = 0,1,2,...,h$, and 2\textsuperscript{nd} statement is true for $|S| = 1,2,...,h+1$

Then, consider the case when $|S| = h+1$

For a particular $E_k$, let

$$S_1 = \{ j \in S \mid (k,j) \text{ is an edge in the dependency graph } G \}$$

$$S_2 = S - S_1 \quad \text{... [ corresponds to mutually independent events]}$$

Note: Since $\text{maxdeg}(G) \leq d$, so $|S_1| \leq d$
Proof: Inductive Case 1 (2)

If \(|S_2| = |S|\), then \(E_k\) is mutually independent of the events \(\neg E_j\) for all \(j\) in \(S\).

In this case:
\[
\Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) = \Pr( E_k ) \leq p \leq 2p
\]

Otherwise, \(|S_2| < |S|\).

In this case, we introduce a notation:
Let \(F_S = \bigcap_{j \in S} \neg E_j\).

Similarly, we define \(F_{S_1}\) and \(F_{S_2}\).
Proof: Inductive Case 1 (3)

Note: $F_S = F_{S_1} \cap F_{S_2}$

So, $\Pr( E_k \mid \bigcap_{j \in s} \neg E_j )$

\[
= \Pr( E_k \mid F_S ) = \frac{\Pr( E_k \cap F_S )}{\Pr(F_S)}
\]

\[
= \frac{\Pr( E_k \cap F_{S_1} \cap F_{S_2} )}{\Pr(F_{S_1} \cap F_{S_2})}
\]

\[
= \frac{\Pr( E_k \cap F_{S_1} \mid F_{S_2}) \Pr(F_{S_2})}{\Pr(F_{S_1} \mid F_{S_2}) \Pr(F_{S_2})}
\]

\[
= \frac{\Pr( E_k \cap F_{S_1} \mid F_{S_2})}{\Pr(F_{S_1} \mid F_{S_2})}
\]
Proof: Inductive Case 1 (4)

From the previous equality, we have

\[ \Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) \]

\[ = \frac{ \Pr( E_k \cap F_{s_1} \mid F_{s_2} ) }{ \Pr( F_{s_1} \mid F_{s_2} ) } \]

\[ \leq \frac{ \Pr( E_k \mid F_{s_2} ) }{ \Pr( F_{s_1} \mid F_{s_2} ) } \]

\[ = \frac{ \Pr( E_k ) }{ \Pr( F_{s_1} \mid F_{s_2} ) } \]

\[ \leq \frac{ p }{ \Pr( F_{s_1} \mid F_{s_2} ) } \quad \text{... (Equation 1)} \]
Proof: Inductive Case 1 (5)

On the other hand, we have

$$\Pr(F_{s_1} | F_{s_2}) = \Pr(\bigcap_{j \in s_1} \neg E_j | \bigcap_{j \in s_2} \neg E_j)$$

$$= 1 - \Pr(\bigcup_{j \in s_1} E_j | \bigcap_{j \in s_2} \neg E_j)$$

$$\geq 1 - \sum_{j \in s_1} \Pr(E_j | \bigcap_{j \in s_2} \neg E_j)$$

$$\geq 1 - \sum_{j \in s_1} 2p \quad \text{... [by induction hypothesis]}$$

$$\geq 1 - 2pd \quad \text{... [since } |s_1| \leq d \text{ ]}$$

$$\geq 1/2 \quad \text{... [since } 4pd \leq 1 \text{ ]}$$
Proof: Inductive Case 1 (6)

So, combining this with Equation 1, we have

\[ \Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) \]

\[ \leq p / \Pr(F_{S_1} \mid F_{S_2}) \leq 2p \]

Thus, 1\textsuperscript{st} statement is true for \(|S| = h+1\)

\[ \Rightarrow \text{This proves Inductive Case 1} \]

It remains to show Inductive Case 2 is true
Proof: Inductive Case 2

Inductive Case 2: Assume 1\textsuperscript{st} and 2\textsuperscript{nd} statement are true for $|S|$ up to $h+1$

Then, consider the case when $|S|=h+2$

$$\Pr(\bigcap_{j \in S} \neg E_j) = \Pr(\bigcap_{j \in \{s_1, s_2, \ldots, s_{h+2}\}} \neg E_j)$$

$$= \prod_{r=1}^{h+2} \Pr(\neg E_{s_r} \mid \bigcap_{t=1}^{r-1} \neg E_{s_t})$$

$$= \prod_{r=1}^{h+2} \left(1 - \Pr(E_{s_r} \mid \bigcap_{t=1}^{r-1} \neg E_{s_t})\right)$$

$$\geq \prod_{r=1}^{h+2} \left(1 - 2p\right) > 0 \quad \text{... [by induction hypothesis]}$$
Conclusion

Thus, 2nd statement is true for $|S| = h+2$

→ This proves Inductive Case 2

• By induction, we can then show that 2\textsuperscript{nd} statement is true for $|S| = n$
• That is, $\Pr(\bigcap_{j \in S} \neg E_j) > 0$ when $|S| = n$

Consequently, we have

$$\Pr(\text{no BAD events}) = \Pr(\bigcap_{j \in S} \neg E_j) > 0$$
Example: Edge-Disjoint Paths

- There are **50** pairs of users in a network system, each pair wants to obtain a dedicated path for communication
  - That is, they do not want their path to share any edge with the path chosen by others
  - Now, we know that each pair has a set of **2000** possible paths to choose, and each such path “crashes” with at most **5** paths in the set of any other pair

**Question:** Can they get a dedicated path?

**Ans.** Yes
**Edge-Disjoint Paths**

In fact, we can show the following based on the Lovasz Local Lemma:

Let $F_j = \text{set of } m \text{ paths pair-} j \text{ can choose}$

**Theorem:** If for all $i \neq j$, each path in $F_i$ "clashes" with no more than $k$ paths in $F_j$, then, when $8nk/m \leq 1$, there exists a way to choose $n$ edge-disjoint paths connecting the $n$ pairs.

How to prove?
Proof

• Let $E_{i,j}$ = event that paths selected by pair-$i$ and pair-$j$ clashes
  $\Rightarrow \Pr(E_{i,j}) \leq \frac{k}{m}$

• Let $G$ = dependency graph of these events

• Since $E_{i,j}$ is dependent only on events $E_{i,x}$ or $E_{y,j}$ $\Rightarrow$ at most $2n$ events

• Now, by setting $p = \frac{k}{m}$ and $d = 2n$,
  $\Pr(E_{i,j}) \leq p$, $\maxdeg(G) \leq d$, and $4pd \leq 1$
  $\Rightarrow$ We can apply LLL, and theorem follows
Lovasz Local Lemma
(General Case)

Next, we describe the general case of LLL
(the proof is extremely similar to the symmetric case):

Theorem: Let $G$ be a dependency graph of a set of BAD events $\{E_1, E_2, \ldots, E_n\}$. Assume that there are $x_1, x_2, \ldots, x_n \in [0,1)$ such that $\Pr(E_i) \leq x_i \prod_{(i,j) \text{ in } G} (1-x_j)$, then

$$\Pr(\text{no BAD events}) \geq \prod_{j=1 \text{ to } n} (1-x_j)$$
Proof

Let \( S = \{s_1, s_2, \ldots \} \) be a subset of \( \{1, 2, \ldots, n\} \)

- The proof is based on induction, where we show two statements are true alternately:

\( \text{(1) } \Pr( E_k \mid \bigcap_{j \in S} \neg E_j) \leq x_k \text{ for all } S, \)

with \( |S| = 0, 1, 2, \ldots, n-1 \)

\( \text{(2) } \Pr(\bigcap_{j \in S} \neg E_j) \geq \prod_{j \in S} (1-x_j) > 0 \)

for all \( S \), with \( |S| = 1, 2, \ldots, n \)
Proof: Base Cases

Base Case 1: 1st statement, \(|S|=0\)
In this case, we have
\[ \Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) = \Pr( E_k ) \leq x_k \] ...
[why??]

\(\Rightarrow\) So this case is true

Base Case 2: 2nd statement, \(|S|=1\)
In this case, we have
\[ \Pr(\bigcap_{j \in S} \neg E_j ) = 1 - \Pr( E_{s_1} ) \geq 1 - x_{s_1} > 0 \]

\(\Rightarrow\) So this case is true
Proof: Inductive Case 1

Inductive Case 1: Assume 1st statement is true for $|S| = 0, 1, 2, ..., h$, and 2nd statement is true for $|S| = 1, 2, ..., h+1$

Then, consider the case when $|S| = h+1$

For a particular $E_k$, let

$$S_1 = \{ j \in S \mid (k, j) \text{ is an edge in the dependency graph } G \}$$

$$S_2 = S - S_1 \quad \text{... [ corresponds to mutually independent events ]}$$
Proof: Inductive Case 1 (2)

If \(|S_2| = |S|\), then \(E_k\) is mutually independent of the events \(\neg E_j\) for all \(j\) in \(S\).

In this case:
\[
\Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) = \Pr( E_k ) \leq x_k
\]

Otherwise, \(|S_2| < |S|\).
In this case:
Let \(F_S = \bigcap_{j \in S} \neg E_j\).

Similarly, we define \(F_{S_1}\) and \(F_{S_2}\).
Proof: Inductive Case 1 (3)

Note: $F_S = F_{S_1} \cap F_{S_2}$

So, $Pr( E_k \mid \bigcap_{j \in S} \neg E_j)$

\[ = Pr( E_k \mid F_S ) = Pr( E_k \cap F_S ) / Pr(F_S) \]

\[ = Pr( E_k \cap F_{S_1} \cap F_{S_2} ) / Pr(F_{S_1} \cap F_{S_2}) \]

\[ = Pr( E_k \cap F_{S_1} \mid F_{S_2} ) Pr(F_{S_2}) / Pr(F_{S_1} \mid F_{S_2}) Pr(F_{S_2}) \]

\[ = Pr( E_k \cap F_{S_1} \mid F_{S_2} ) / Pr(F_{S_1} \mid F_{S_2}) \]
Proof: Inductive Case 1 (4)

From the previous equality, we have

\[ \Pr( E_k \mid \bigcap_{j \in S} \neg E_j ) = \frac{\Pr( E_k \cap F_{S_1} \mid F_{S_2} )}{\Pr( F_{S_1} \mid F_{S_2} )} \leq \frac{\Pr( E_k \mid F_{S_2} )}{\Pr( F_{S_1} \mid F_{S_2} )} = \frac{\Pr( E_k )}{\Pr( F_{S_1} \mid F_{S_2} )} \leq x_k \prod_{(k,j) \in G} (1-x_j) / \Pr( F_{S_1} \mid F_{S_2} ) \]  

... (Equation 1)
Now, we label the element of $S_1$ by $\{y_1, y_2, \ldots, y_r\}$:

$$\Pr(F_{S_1} \mid F_{S_2}) = \Pr(\bigcap_{j \in S_1} \neg E_j \mid \bigcap_{j \in S_2} \neg E_j)$$

$$= \prod_{t=1}^{r} \Pr(\neg E_{y_t} \mid \bigcap_{v=1}^{t-1} \neg E_{y_v} \cap \bigcap_{j \in S_2} \neg E_j) \quad \text{**}$$

$$= \prod_{t=1}^{r} \left(1 - \Pr(E_{y_t} \mid \bigcap_{v=1}^{t-1} \neg E_{y_v} \cap \bigcap_{j \in S_2} \neg E_j)\right)$$

$$\geq \prod_{t=1}^{r} \left(1 - x_{y_t}\right) \quad \ldots \quad \text{[by induction hypothesis]}$$

$$\geq \prod_{(k,j) \in G} (1-x_j) \quad \ldots \quad \text{[why??]}$$

**Proof: Inductive Case 1 (5)**
Proof: Inductive Case 1 (6)

So, combining this with Equation 1, we have

\[
\Pr( E_k | \bigcap_{j \in S} \neg E_j )
\]
\[
\leq x_k \prod_{(k,j) \in G} (1-x_j) / \Pr(F_{S_1} | F_{S_2}) \leq x_k
\]

Thus, 1\textsuperscript{st} statement is true for \(|S| = h+1\)

\(\Rightarrow\) This proves Inductive Case 1

It remains to show Inductive Case 2 is true
Proof: Inductive Case 2

Inductive Case 2: Assume 1st and 2nd statement are true for $|S|$ up to $h+1$

Then, consider the case when $|S| = h+2$

$$
\text{Pr}(\bigcap_{j \in S} \neg E_j) = \text{Pr}(\bigcap_{j \in \{s_1, s_2, ..., s_{h+2}\}} \neg E_j)
$$

$$
= \prod_{r=1}^{h+2} \text{Pr}(\neg E_{s_r} \mid \bigcap_{t=1}^{r-1} \neg E_{s_t})
$$

$$
= \prod_{r=1}^{h+2} (1 - \text{Pr}(E_{s_r} \mid \bigcap_{t=1}^{r-1} \neg E_{s_t}))
$$

$$
\geq \prod_{r=1}^{h+2} (1 - x_{s_r}) \quad \text{[by induction hypothesis]}
$$
Conclusion

Since \( \prod_{r=1}^{h+2} (1 - x_{sr}) = \prod_{j \in S} (1-x_j) > 0 \)

Thus, 2nd statement is true for \( |S| = h+2 \)

\( \Rightarrow \) This proves Inductive Case 2

By induction, we can then show that 2nd statement is true for \( |S| = n \)

Consequently, we have

\[ \Pr(\text{no BAD events}) = \Pr(\bigcap_{j \in \{1,2,\ldots,n\}} \neg E_j) \]

\[ \geq \prod_{j=1}^{n} (1-x_j) > 0 \]
Lovasz Local Lemma
(Symmetric Case -- revisited)

The general case can immediately improve the symmetric case by replacing the condition $4pd \leq 1$ to $ep(d+1) \leq 1$, so that we can apply it in more situations.

The proof is by setting all $x_i = 1/(d+1)$

⇒ Then, we can show that

$$\Pr(E_i) \leq p \leq x_i \prod_{(i,j) \text{ in } G} (1-x_j) \quad \text{... [how?] }$$

so that we can apply the General Case

(Left as an Exercise)