CS5314
Randomized Algorithms

Lecture 24: Markov Chains
(Parrondo’s Paradox)
Objectives

• Introduce Parrondo’s Paradox
  - named after a Spanish physicist Juan Parrondo (1964--)

• The Paradox describes an interesting example of two games $A$ and $B$, such that if we play any one of them (say $A$) in the long run, we will be losing, but ...
  if each time, we choose $A$ or $B$ to play with equal probability, then we may be winning in the long run !!!
Game A

• Game A is very simple: We have a biased coin, such that it comes up head with probability 0.49, it comes up tail with probability 0.51.

• In the game, you will win $1 if head comes up, and lose $1 otherwise...

Question: If you can play Game A again and again, would you like to do so?
Game B

- Game B is a bit complicated: We have two biased coins. Depending on the current money you have, we will choose which of the biased coins to use.
  
  1st coin: (when your money not multiple of 3)
  - it comes up head with probability 0.74,
  - it comes up tail with probability 0.26

  2nd coin: (when your money is multiple of 3)
  - it comes up head with probability 0.09,
  - it comes up tail with probability 0.91
Game B (cont)

- Again, in this game, you will win $1 if head comes up, and lose $1 otherwise.
- In general, Game B can be stated as: we win with probability $p_{12}$ if our money is not a multiple of 3, and we win with probability $p_3$ otherwise.

**Question:** If you can play Game B again and again, would you like to do so?

**Idea.** We play Game B only if it is more likely to win $3 before losing $3 ... [why?]
The previous idea can be modeled by the following Markov chain:

Markov Chain for Game B
Markov Chain for Game B (2)

- Let $z_j$ be the probability of winning $3$ before losing $3$ when starting at state $j$.
- Based on this definition, we have:
  
  $z_{-3} = 0$ and $z_3 = 1$

Also, we have:

\[
\begin{align*}
    z_{-2} &= (1-p_{12}) z_{-3} + p_{12} z_{-1} \\
    z_{-1} &= (1-p_{12}) z_{-2} + p_{12} z_0 \\
    z_0 &= (1-p_{3}) z_{-1} + p_{3} z_1 \\
    z_1 &= (1-p_{12}) z_0 + p_{12} z_2 \\
    z_2 &= (1-p_{12}) z_1 + p_{12} z_3
\end{align*}
\]
Markov Chain for Game B (3)

• Since $p_{12}$ and $p_3$ are given, the previous system has 7 equations and 7 unknowns, so that it can be solved easily.

• In particular, we have:

$$z_0 = \frac{p_3 p_{12}^2}{((1-p_3)(1-p_{12})^2 + p_3 p_{12}^2)}$$

• By definition, $z_0 = \text{prob of winning } 3\text{ before losing } 3, \text{ when starting with } 0$

$\Rightarrow$ if $z_0 > 0.5$, we should play Game B again and again; else, we should not ...
**Markov Chain for Game B (4)**

In our example,

\[ p_{12} = 0.74 \quad \text{and} \quad p_3 = 0.09 \]

Thus,

\[ z_0 = \frac{p_3 p_{12}^2}{(1-p_3)(1-p_{12})^2 + p_3 p_{12}^2} \]

\[ = \frac{(0.09)(0.74)^2}{(0.91)(0.26)^2 + (0.09)(0.74)^2} \]

\[ = 0.049284 / 0.1108 \]

\[ < 0.5 \]

So, Game B is also a bad choice for us ...
After going through the above analysis, we know that neither Game A nor Game B is a good choice to play in the long run ...

Now, we have Game C as follows:
1. Flip a fair coin.
2. If head comes up, we play Game A. Else, we play Game B

That means, after a game, we will still either win $1 or lose $1
Question: If you can play Game C again and again, would you like to do so?

Intuition: Roughly speaking, if we play Game C again and again, we will play Game A and play Game B each 50% of the time... Both A and B are not favorable for us ... So, it seems like Game C is losing ...

But, is it really true??
Game C (cont)

• Let us analyze whether we should play Game C using the same idea as before

• First, let $q_{12}$ be the probability of winning when our money is not a multiple of 3, and let $q_3$ be the probability of winning when our money is a multiple of 3

• Again, we play Game C only if it is more likely to win $3 before losing $3
The previous idea can be modeled by the following Markov chain:
Thus, the probability of winning $3 in Game C before losing $3, when we start with $0, is:

\[ z = \frac{q_3 q_{12}^2}{(1-q_3)(1-q_{12})^2 + q_3 q_{12}^2} \]

**Question:** What are the values of \( q_{12} \) and \( q_3 \) in our example??

**Ans.**  
\[ q_{12} = 0.5 \times 0.49 + 0.5 \times 0.74 = 0.615 \]  
\[ q_3 = 0.5 \times 0.49 + 0.5 \times 0.09 = 0.29 \]
Thus,

\[ z = \frac{q_3 q_{12}^2}{(1-q_3)(1-q_{12})^2 + q_3 q_{12}^2} \]

\[ = \frac{0.29 \cdot 0.615^2}{(0.71)(0.385)^2 + (0.29)(0.615)^2} \]

\[ = \frac{0.10968525}{0.214925} > 0.5 \]

So, Game C is a good choice for us !!!
How does Game A help us?

This is Game B

This is Game C
Final Remarks

I hope you like and enjoy the course
(... Sorry that I haven’t enough time to cover all the interesting topics in the textbook

⇒ I hope that you can find your spare time to read the uncovered chapters...)

Thanks Joyce for being a wonderful tutor

Thanks all of you for coming to the class!

Finally, **Good Luck!** in the exam ^__^