CS5371
Theory of Computation

Lecture 1: Mathematics Review I
(Basic Terminology)
Objectives

• Unlike other CS courses, this course is a MATH course...
• We will look at a lot of definitions, theorems and proofs
• This lecture: reviews basic math notation and terminology
  – Set, Sequence, Function, Graph, String...
• Also, common proof techniques
  – By construction, induction, contradiction
Set

- A set is a group of items
- One way to describe a set: list every item in the group inside { }
  - E.g., { 12, 24, 5 } is a set with three items
- When the items in the set has trend: use ...
  - E.g., { 1, 2, 3, 4, ... } means the set of natural numbers
- Or, state the rule
  - E.g., { n | n = m^2 for some positive integer m } means the set { 1, 4, 9, 16, 25, ... }
- A set with no items is an empty set denoted by {} or ∅
Set

- The order of describing a set does not matter
  - \{ 12, 24, 5 \} = \{ 5, 24, 12 \}
- Repetition of items does not matter too
  - \{ 5, 5, 5, 1 \} = \{ 1, 5 \}
- Membership symbol \( \in \)
  - \( 5 \in \{ 12, 24, 5 \} \) \quad 7 \notin \{ 12, 24, 5 \}
Set (Quick Quiz)

• How many items are in each of the following set?
  - \{ 3, 4, 5, \ldots, 10 \}
  - \{ 2, 3, 3, 4, 4, 2, 1 \}
  - \{ 2, \{2\}, \{\{2\}\} \}
  - \emptyset
  - \{ \emptyset \}
Given two sets $A$ and $B$

- we say $A \subseteq B$ (read as $A$ is a subset of $B$) if every item in $A$ also appears in $B$
  - E.g., $A =$ the set of primes, $B =$ the set of integers

- we say $A \subsetneq B$ (read as $A$ is a proper subset of $B$) if $A \subseteq B$ but $A \neq B$

**Warning:** Don’t be confused with $\in$ and $\subseteq$

- Let $A = \{1, 2, 3\}$. Is $\emptyset \in A$? Is $\emptyset \subseteq A$?
Union, Intersection, Complement

Given two sets $A$ and $B$

- $A \cup B$ (read as the **union** of $A$ and $B$) is the set obtained by combining all elements of $A$ and $B$ in a single set
  - E.g., $A = \{ 1, 2, 4 \}$ $B = \{ 2, 5 \}$
    $A \cup B = \{ 1, 2, 4, 5 \}$

- $A \cap B$ (read as the **intersection** of $A$ and $B$) is the set of common items of $A$ and $B$
  - In the above example, $A \cap B = \{ 2 \}$

- $\bar{A}$ (read as the **complement** of $A$) is the set of items under consideration not in $A$
Set

- The **power set** of $A$ is the set of all subsets of $A$, denoted by $2^A$
  - E.g., $A = \{ 0, 1 \}$
    \[ 2^A = \{ \{\}, \{0\}, \{1\}, \{0,1\} \} \]
  - How many items in the above power set of $A$?
- If $A$ has $n$ items, how many items does its power set contain? Why?
Sequence

• A sequence of items is a list of these items in some order
• One way to describe a sequence: list the items inside ( )
  - ( 5, 12, 24 )
• Order of items inside ( ) matters
  - ( 5, 12, 24 ) \neq ( 12, 5, 24 )
• Repetition also matters
  - ( 5, 12, 24 ) \neq ( 5, 12, 12, 24 )
• Finite sequences are also called tuples
  - ( 5, 12, 24 ) is a 3-tuple
  - ( 5, 12, 12, 24 ) is a 4-tuple
Sequence

Given two sets $A$ and $B$

- The **Cartesian product** of $A$ and $B$, denoted by $A \times B$, is the set of all possible 2-tuples with the first item from $A$ and the second item from $B$
  - E.g., $A = \{1, 2\}$ and $B = \{x, y, z\}$
    $$A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$$

- The **Cartesian product** of $k$ sets, $A_1, A_2, \ldots, A_k$, denoted by $A_1 \times A_2 \times \cdots \times A_k$, is the set of all possible $k$-tuples with the $i^{th}$ item from $A_i$
Functions

- A function takes an input and produces an output.
- If \( f \) is a function, which gives an output \( b \) when input is \( a \), we write
  \[ f(a) = b \]
- For a particular function \( f \), the set of all possible input is called \( f \)’s domain.
- The outputs of a function come from a set called \( f \)’s range.
Functions

• To describe the property of a function that it has domain $D$ and range $R$, we write
  \[ f : D \rightarrow R \]

• E.g., The function add (to add two numbers) will have an input of two integers, and output of an integer
  - We write: \( \text{add}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \)
Functions (Quick Quiz)

• **Guess:** What does the following function DOW do?
  - DOW(9,12) = 2
  - DOW(9,13) = 3
  - DOW(9,17) = 7
  - DOW(9,18) = 1

• What are the domain and the range of DOW?
Graphs

- A graph is a set of points with lines connecting some of the points.
- Points are called \textit{vertices}, lines are called \textit{edges}.
- E.g.,
Graphs

• The number of edges at a particular vertex is the **degree** of the vertex.

• In the previous example, 3 vertices have degree = 2.

• A graph can be described by telling what are its vertices, and what are its edges. Formally, a graph $G$ can be written as $G = (V, E)$, where $V$ is the set of vertices, and $E$ is the set of edges.
Graphs

• We say a graph $G$ is a subgraph of $H$ if vertices of $G$ are a subset of the vertices of $H$, and all edges in $G$ are the edges of $H$ on the corresponding vertices.

Graph $H$

Subgraph $G$ shown darker
Graphs

- A **path** is a sequence of vertices connected by edges.
- If every two nodes have a path between them, the graph is **connected**.
- A **cycle** is a path that starts and ends at the same vertex.
- A **tree** is a connected graph with no cycles.
Graphs (Quick Quiz)

• Is the following graph connected?
• Is it a tree?
• Are there any cycles?
• How about the darker subgraph?
Directed Graphs

- If lines are replaced by arrows, the graph becomes **directed**.
- The number of arrows pointing into a vertex is called **in-degree** of the vertex.
- The number of arrows pointing from a vertex is called **out-degree** of the vertex.
- A **directed path** is a path from one vertex to the other vertex, following the direction of the “arrows”.
Directed Graphs

• Is there a directed path from a to b?
Strings

• An alphabet = a set of characters
  - E.g., The English Alphabet = \{A,B,C,...,Z\}
• A string = a sequence of characters
• A string over an alphabet \( \Sigma \)
  - A sequence of characters, with each character coming from \( \Sigma \)
• The length of a string \( w \), denoted by \(|w|\), is the number of characters in \( w \)
• The empty string (written as \( \varepsilon \)) is a string of length 0
Strings

Let $w = w_1w_2...w_n$ be a string of length $n$

- A substring of $w$ is a consecutive subsequence of $w$ (that is, $w_iw_{i+1}...w_j$ for some $i \leq j$)
- The reverse of $w$, denoted by $w^R$, is the string $w_n...w_2w_1$
- A set of strings is called a language
Next time

- Common Proof Techniques
- Part I: Automata Theory