CS 5319
Advanced Discrete Structure

Permutations and Combinations I
Outline

• Notation
• Rules of Sum and Product
• Permutations
• Combinations
• Distribution of Objects
* Stirling’s Formula
Notation

• Selection and arrangement of objects appear in many places
  ➔ we often want to compute # of ways to select or arrange the objects

• Ex:
  1. How many ways to select 2 people from 5 candidates?
  2. How many ways to arrange 5 books on the bookshelf?
Notation

• In most textbooks, we use the word combination $\Leftrightarrow$ selection

Definition: An $r$-combination of $n$ objects is an unordered selection of $r$ of these objects

• Ex: \{c, d\} is a 2-combination of the 5 objects \{a, b, c, d, e\}
Notation

• In most textbooks, we use the word permutation ⇔ arrangement

Definition: An \( r \)-permutation of \( n \) objects is an ordered arrangement of \( r \) of these objects

• Ex: \( cbade \) is a \( 5 \)-permutation of the 5 objects \{ a, b, c, d, e \}
Notation

• Further, we use the following notation:

The notation $C(n,r)$ denotes the number of $r$-combination of $n$ distinct objects

The notation $P(n,r)$ denotes the number of $r$-permutation of $n$ distinct objects

• What are the values of $C(n,n)$, $C(n,1)$, $C(3,2)$, and $P(3,2)$?
Rules of Sum and Product

• Suppose we have

  5 Roman letters A, B, C, D, E

  and 3 Greek letters α, β, γ

• How many ways to select two letters, one from each group?

• How many ways to select one letter that is either a Roman or a Greek letter?
Rules of Sum and Product

• In general, if
  one event can occur in $m$ ways and
  another event can occur in $n$ ways,

**Rule of Product:** There are $m \times n$ ways that these two events can occur together

**Rule of Sum:** There are $m + n$ ways that one of these two events can occur
Rules of Sum and Product

Ex: Suppose there are

5 Chinese books, 7 English books, and
10 French books

- How many ways to choose 2 books of different languages from them?

- Ans: $5 \times 7 + 5 \times 10 + 7 \times 10 = 155$
Rules of Sum and Product

Ex:

Why are the following formulas correct?

1. $P(n,r) = P(r,r) \times C(n,r)$
2. $P(n,n) = P(n,r) \times P(n-r,n-r)$
3. $C(n,r) = C(n-1,r-1) + C(n-1,r)$
Permutations
Permutations

• We can show that

\[ P(n,r) = n \times (n-1) \times \ldots \times (n-r+1) \]
\[ = \frac{n!}{(n-r)!} \]

*Proof*: \( P(n,r) = \# \) ways to get \( r \) of \( n \) objects in some order.

There are \( n \) ways to get the first object, \( n-1 \) ways to get the second object, ... , \( n-r+1 \) ways to get the last object.

\( \rightarrow \) Result follows from rule of product.
Permutations

Ex: How many ways can $n$ people stand to form a ring?

The above are considered to be the same (as relative order is the same)
Permutations

- Suppose we have $n$ objects which are not all distinct, where
  
  $q_1$ objects are of the first kind,
  
  $q_2$ objects are of the second kind,
  
  ...
  
  $q_t$ objects are of the $t$th kind.

$\Rightarrow$ # of $n$-permutation of these objects is:

$$\frac{n!}{q_1! \cdot q_2! \cdot \ldots \cdot q_t!}$$
Permutations

Ex:
Suppose we have 5 dashes and 8 dots

⇒ $13! / (5!8!)$ ways to arrange them

• If we can only use 7 symbols from them, how many different arrangements?

Ans: $7! / (5!2!) + 7! / (4!3!) + 7! / (3!4!) + 7! / (2!5!) + 7! / (1!6!) + 7! / 7! = 120$
Permutations

Ex: How to show that

$$(k!)! \text{ is divisible by } k! \cdot (k-1)! ?$$

- Consider the permutation of $k!$ objects, where
  - $k$ are of the first kind,
  - $k$ are of the second kind,
  - …
  - $k$ are of the $(k-1)!$ th kind.
Permutations

• Suppose we have \( n \) distinct objects, each with unlimited supply

• The # of ways to arrange \( r \) objects from them is:

\[
\binom{n}{r}
\]

\( n^r \)
Permutations

Ex: Consider the numbers between 1 and \(10^{10}\).

• How many of them contain the digit 1?
• How many of them do not?

Ans: \(9^{10} - 1\) of them do not, the others do.
Permutations

Ex: Consider all $n$-digit binary strings.

• How many contain even number of 0’s?
  
  Ans: Half of them (by symmetry)

Ex: Consider all $n$-digit quaternary strings.

• How many contain even number of 0’s?
  
  Ans: $2^n + (4^n - 2^n) / 2$ (how to get this?)
Combinations
Combinations

• Recall that

\[ P(n,r) = P(r,r) \times C(n,r) \]

Thus we have:

\[
C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)! r!}
\]

• Immediately, we also have

\[ C(n,r) = C(n,n-r) \]
Combinations

Ex: Consider a decagon (10-sided) where no three diagonals meet at a point.

• How many line segments are the diagonals divided by their intersections?

In case of a pentagon, there will be 15 line segments
Combinations

Ex:

Five pirates have discovered a treasure box. They decide to keep that in a locked room so that all the locks can be opened if and only if 3 or more pirates are present.

• How can they do so? How many locks needed? (Each pirate can possess keys to different locks)
Combinations

Ex:

In how many ways can we select three numbers from 1, 2, …, 300 such that their sum is divisible by 3?

– When the sum of three numbers is divisible by 3, what special property do they have?

• Ans: $C(100,3) + C(100,3) + C(100,3) + 100^3$
Combinations

• Suppose we have \( n \) distinct kinds of objects, each with unlimited supply
• The # of ways to select \( r \) objects from them is:

\[
C(n+r-1, \ r)
\]

• How to prove it ?
Combinations

Ex:

When three indistinguishable dice are thrown, how many outcomes are there?

- Ans: $C(6+3-1,3) = 56$
Combinations

Ex:

Out of a number of $100, $200, $500, $1000 bills, how many ways can six bills be selected?

• Ans: \( C(4+6-1, 6) = 84 \)
Combinations

• Suppose we have \( n \) objects which are not all distinct, where

\[
q_1 \text{ objects are of the first kind,}
\]
\[
q_2 \text{ objects are of the second kind,}
\]
\[
\ldots
\]
\[
q_t \text{ objects are of the } t \text{ th kind.}
\]

\[\Rightarrow \text{ number of ways to select one or more of these objects from them is:}\]
\[
(q_1+1)(q_2+1) \ldots (q_t+1) - 1
\]
Combinations

Ex:

How many divisors does 1400 have?

• Ans:

Since $1400 = 2^3 \times 5^2 \times 7$, the number of divisors of 1400 is

$(3+1) \times (2+1)(1+1) = 24$
Combinations

Ex:

For \( n \) given weights, what is the greatest number of different amounts that can be made up by the combinations of these weights?

To weigh things with integral weight between 1 and 100, how many weights do we need?
Combinations

Ex:

What is the greatest number of different amounts that can be weighed using $n$ weights and a balance?

To weigh things with integral weight between 1 and 100, how many weights do we need?