1. Prove the following identity using combinatorial arguments:

\[ C(n + 1, m) = C(n, m) + C(n - 1, m - 1) + C(n - 2, m - 2) + \cdots + C(n - m, 0) \quad \text{for } m \leq n \]

2. Prove the following identity using combinatorial arguments:

\[ n \times C(n - 1, r) = (r + 1) \times C(n, r + 1). \]

3. (a) Using combinatorial arguments, prove that \((2n)!/2^n\) is an integer.

(b) Prove that \((3n)!/(2^n \times 3^n)\) is an integer. (Hint: 2 and 3 are relatively prime.)

4. There is a new fast food restaurant in town. If you have \(k\) dollars to spend (\(k\) is an integer), their menu has exactly \(k^2 + 1\) different sets with cost \(k\) that you may choose.†

Four identical quadruples come to this restaurant, and they are going to spend exactly 4 dollars in total. How many different configurations can these quadruples try?

For instance, one possible configuration is: three of them spend 0 dollars, each getting the set that costs 0 dollars, and the last one spends 4 dollars, choosing one set from the 17 sets that costs 4 dollars. Note that the quadruples are identical, so that we do not care which three of them get the sets costing 0 dollars.

5. Suppose that no three of the diagonals of a convex \(n\)-gon meet at the same point inside of the \(n\)-gon. Triangles will be formed with the sides of made up of the sides of the \(n\)-gon, the diagonals, or segments of the diagonals. How many different triangles are there?

For instance, when \(n = 4\), there will be 8 triangles.

6. (Challenging: No marks) How many permutations of the integers 1, 2, \ldots, \(n\) are there such that every integer is followed by (but not necessarily immediately followed by) an integer which differs from it by 1?

For example, with \(n = 4\), 1432 is an acceptable permutation but 2431 is not.

7. (Challenging: No marks) Consider a parenthesis sequence with \(n\) (‘s and \(n\) )’s. Such a sequence is called valid if for any \(m \leq n\), the number of (‘s in the first \(m\) symbols is at least the number of )’s in the first \(m\) symbols. Otherwise, a sequence is called invalid. How many valid sequences are there?

For instance, \((()\) or \((())\) are both valid, while \(((()\) and \((())\) are both invalid.

\textit{Hint}: Given an invalid sequence, we are going to obtain a modified sequence as follows: (1) Suppose \(k\) is the smallest number such that there are more ) than ( in the first \(k\) symbols. (2) Flip the sequence starting from the \(k + 1\)th symbol. What is so special about the modified sequence? How many are there?

†That means, if you have no money, you still have something to eat!!