CS4311
Design and Analysis of Algorithms

Lecture 7:
Lower Bound of Comparison Sorts
About this lecture

• Lower bound of any comparison sorting algorithm
  - applies to insertion sort, selection sort, merge sort, heapsort, quicksort, ...
  - does not apply to counting sort, radix sort, bucket sort

• Based on Decision Tree Model
Comparison Sort

• Comparison sort only uses comparisons between items to gain information about the relative order of items

• It’s like the elements are stored in boxes, and we can only pick two boxes at a time to compare which one is larger, without knowing their values
Worst-Case Running Time

Merge sort and heapsort are the “smartest” comparison sorting algorithms we have studied so far:

worst-case running time is $\Theta(n \log n)$

Question: Do we have an even smarter algorithm? Say, runs in $o(n \log n)$ time?

Answer: No! (main theorem in this lecture)
Theorem: Any comparison sorting algorithm requires $\Omega(n \log n)$ comparisons to sort $n$ distinct items in the worst case.

Corollary: Any comparison sorting algorithm runs in $\Omega(n \log n)$ time in the worst case.

Corollary: Merge sort and Heapsort are (asymptotically) optimal comparison sorts.
Proof of Lower Bound

The main theorem only counts comparison operations, so we may assume all other operations (such as moving items) are for free.

Consequently, any comparison sort can be viewed as performing in the following way:

1. Continuously gather relative ordering information between items
2. In the end, move items to correct positions

We use the above view in the proof
Consider the following algorithm to sort 3 items A, B, and C:

Step 1: Compare A with B
Step 2: Compare B with C
Step 3: Compare A with C

Afterwards, decide the sorting order of the 3 items.
The previous algorithm always use 3 comparisons, and can sort the 3 items.

In particular, the comparisons used in different inputs (i.e., permutations) can be captured in a decision tree, as shown in the next slide:
A : B

B : C

A : C

result of decision

sorting order decided

impossible case
A cleverer algorithm may sort the 3 items, sometimes, using at most 2 comparisons:

Step 1: Check if $A > B$
Step 2: Check if $B > C$
Step 3: Compare $A$ with $C$ if the result in Steps 1 and 2 are different

Afterwards, decide the sorting order

• Then, the decision tree becomes ...
decision

A : B

B : C

A : C

B : C

result of decision

sorting order decided
Properties of Decision Tree

In general, assume the input has \( n \) items

Then, for \textbf{ANY} comparison sort algorithm:

- Each of the \( n! \) permutations corresponds to a distinct leaf in the decision tree
- The \textbf{height} of the tree is the worst-case \# of comparisons for any input

Question: What can be the \textbf{height} of the decision tree of the cleverest algorithm?
Lower Bound on Height

- There are $n!$ leaves [for any decision tree]
- Degree of each node is at most 2
- Let $h =$ node-height of decision tree

So, $n! = \text{total # leaves} \leq 2^h$

$\Rightarrow h \geq \log (n!) = \log n + \log (n-1) + \ldots$

$\geq \log n + \ldots + \log (n/2)$

$\geq (n/2) \log (n/2) = \Omega(n \log n)$

We can also use Stirling's approximation:

$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1+\Theta(1/n))$
Proof of Lower Bound

Conclusion:

worst-case # of comparisons
= node-height of the decision tree
= \Omega(n \log n) \ [for any decision tree]

\Rightarrow \text{Any comparison sort, even the cleverest one, needs } \Omega(n \log n) \text{ comparisons in the worst case}