1. Let $G = (V, E)$ be a connected, undirected graph. An articulation point of $G$ is a vertex whose removal will disconnect $G$. Suppose we perform DFS on $G$, and let $T$ be the resulting DFS tree. We are going to find all articulation points of $G$ based on $T$.

(a) (25%) Prove that the root of $T$ is an articulation point of $G$ if and only if it has at least two children in $T$.

Hint: ($\Leftarrow$) If the root of $T$ has two children, $c_1$ and $c_2$. Can they be connected by a path in $G$ without the root? (White-path theorem may be useful.)

(b) (25%) Let $v$ be a non-root vertex of $T$. Prove that $v$ is an articulation point of $G$ if and only if $v$ has a child $s$ such that there is no back edge from $s$ or any descendant of $s$ to a proper ancestor of $v$.

Hint: ($\Rightarrow$) If for each child $s$ of $v$, there is a back edge from $s$ or its descendant to a proper ancestor of $v$, can you show that every neighbor of $v$ is connected to parent($v$)? (Be careful!!! Some neighbors of $v$ may not be children of $v$.) In this case, can $v$ be an articulation point?

Hint: ($\Leftarrow$) If all back edges from $s$ or from its descendant do not point to any proper ancestor of $v$, where can they point to? Can you show that if $v$ is removed, $s$ and parent($v$) are disconnected?

(c) (25%) Let $low[v] = \min \{ d(v), d(w) \}$ such that $(u, w)$ is a back edge from some descendant $u$ of $v$. Show how to compute $low[v]$ for all vertices $v \in V$ in $O(|E|)$ time.

(Hint: Recall $T$ is the DFS tree. Suppose for a node $v$, its children are $c_1, c_2, \ldots, c_k$.
Can you show any relationship among $low[v]$ and $low[c_1], low[c_2], \ldots, low[c_k]$?)

(d) (25%) Show how to compute all articulation points in $O(|E|)$ time.

2. (Bonus: 10%) A directed graph is said to be semi-connected if for all pairs of vertices $u$ and $v$, we have $u \rightsquigarrow v$, or $v \rightsquigarrow u$, or both. (The notation $u \rightsquigarrow v$ means $u$ can reach $v$ by a directed path.)

(a) (5%) Suppose $G$ is a directed acyclic graph with $n$ vertices, and suppose we have performed a topological sort on $G$. Let $v_i$ denote the $i$th vertex in the topological sort order.

Show that $G$ is semi-connected if and only if there is an edge $(v_i, v_{i+1})$ for all $i = 1, 2, \ldots, n - 1$.

(b) (5%) Suppose $G$ is a general directed graph (which may contains cycle). Give an $O(|V|+|E|)$-time algorithm to check if $G$ is semi-connected. Show that your algorithm is correct.

(Hint: Finding SCC, then topological sort on component graph.)

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5 Q2 is a bonus question. Total mark is calculated by: Q1 $\times$ (100% + Q2).