(30pts) 1. For the following statements, mark a circle if it is true, and mark a cross otherwise.

(a) Let a r.v. $X \sim b(50, 0.6)$, then $Var(X) = 12$.

(b) If the p.m.f. of a geometric distribution is $f(x) = \frac{3}{4^x}$, $x = 1, 2, 3, \ldots$, then $E(X) = \frac{3}{4}$.

(c) A Poisson distribution $f(x) = e^{-2x}/x!$, $x = 0, 1, 2, \ldots$, has variance 2.

(d) Let $X$ be a r.v. of negative binomial distribution whose p.m.f. is $P(X = x) = \binom{x - 1}{4} \frac{1}{2^x}$, $x = 5, 6, 7, \ldots$, then $E(X) = 10$ and $Var(X) = 10$.

(e) Let $X$ be a r.v. of hypergeometric distribution whose p.m.f. is $P(Y = y) = \frac{\binom{12}{y} \binom{18}{15-y}}{\binom{30}{15}}$, $y = 0, 1, \cdots, 15$, then $E(Y) = 6$ and $Var(Y) = \frac{54}{29}$.

(f) In a certain random experiment, let $A$ and $B$ be two events such that $P(A) = 0.8$, $P(B) = 0.5$, and $P((A \cup B)') = 0.1$. Then $A$ and $B$ are independent events.

(g) In a lottery, an integer is selected at random from 0 to 9, inclusive. Let $X$ be the integer selected on a particular day. Then $X$ has a uniform distribution with $E(X) = 4.5$ and $Var(X) = 8.25$.

(h) Let $X \sim b(30, 0.8)$, then $E(X) = 24$ and $Var(X) = 4.8$.

(i) Let $X$ have a geometric distribution with mean $\frac{4}{3}$, then $P(X > 5) = \frac{1}{1024}$.

(j) Let $X$ have an exponential distribution with variance 9, then its p.m.f. is $f(x) = \frac{1}{3}e^{-x/3}$, $x \geq 0$.

(k) Let $X$ have an exponential distribution with mean $\theta > 0$, then for any $a, b > 0$, we have $P(X > a) \times P(X > b) = P(X > a + b)$.

(l) Let $A, B$ be two independent events, then $P(A \cap B) = P(A)P(B)$.

(m) A r.v. $X$ whose p.d.f. $f(x) = \frac{\pi}{2} \sin(\pi x)$, $0 < x < 1$ has the median $\pi_{0.50} = \frac{1}{2}$.

(n) Let $Z \sim N(0, 2)$, then $M_Z(t) = e^{t^2}$, $t > 0$.

(o) The p.m.f. of a Poisson distribution with variance 4 has the moment-generating function $M(t) = e^{4(e^t - 1)}$. 
(20pts) 2. Choose the best (unique) solution for each of the following problems.

(a) Let $X$ have a density function $f(x) = \frac{1}{2^x}xe^{-x/2}$, $x \geq 0$, then the variance of $X$ is
   (1) 8, (2) 4, (3) $\frac{1}{2}$, (4) $\frac{1}{4}$, (5) none

(b) Let $X$ have a density function $f(x) = \frac{1}{2}e^{-x/2}$, $x \geq 0$, then the median of $X$ is
   (1) $2\ln 2$, (2) $4\ln 2$, (3) $e^{-2}$, (4) $e^{-4}$, (5) none

(c) Let $X$ be a gamma distribution with p.d.f. $f(x) = \frac{3}{2}xe^{-x/3}$, $0 \leq x \leq \infty$, then the variance of $X$ is
   (1) 9, (2) 12, (3) 18, (4) 24, (5) none

(d) Let $X$ be a gamma distribution with p.d.f. $f(x) = \frac{1}{16}x^2e^{-x/2}$, $0 \leq x \leq \infty$, then moment-generating function is
   (1) $\frac{1}{1-3t}$, (2) $\frac{1}{(1-3t)^2}$, (3) $\frac{1}{1-2t}$, (4) $\frac{1}{1-2t^2}$, (5) none

(e) Let $X$ be a $\chi^2$ distribution whose moment-generating function $\phi(t) = \frac{1}{(1-2t)^3}$, $t < \frac{1}{2}$, then the variance $\text{Var}(X)$ is
   (1) 2, (2) 3, (3) 6, (4) 12, (5) none

(f) Let $X$ be a $\chi^2$ distribution whose moment-generating function $\phi(t) = \frac{1}{(1-2t)^3}$, $t < \frac{1}{2}$, then the third quartile of $X$ is
   (1) 0.75, (2) $2 \ln(2)$, (3) $4 \ln(2)$, (4) 0.25, (5) none

(g) Let $X \sim N(1, 4)$, then the moment-generating function of $X$ is
   (1) $e^{t+4t^2}$, (2) $e^{t+2t^2}$, (3) $\frac{1}{1-4t^2}$, (4) $\frac{1}{1-4t^2}$, (5) none

(h) Let $X \sim N(4, 4)$, then the median of $X$ is
   (1) 4, (2) 3, (3) 2, (4) 1, (5) none

(i) Let $Z \sim N(0, 1)$ and define $Y = Z^2$, then $\text{Var}(Y)$ equals
   (1) 1, (2) 2, (3) 3, (4) 4, (5) none

(j) Let $X \sim N(3, 4)$ and define $Y = (X - 3)/2$, then $\text{Var}(Y^2)$ equals
   (1) 1, (2) 2, (3) 3, (4) 4, (5) none

1 2 3 4 4 3 2 1 2 2
(20pts) 3. Fill the following blanks.

(a) Let $A$ and $B$ be independent events with $P(A) = 0.5$ and $P(B) = 0.8$, then

\[
P(A \cap B) = 0.4
\]
\[
P(A \cup B) = 0.9
\]

(b) Let $X \sim b(100, 0.8)$, then

\[
E(X) = 80
\]
\[
Var(X) = 16
\]
\[
M(t) = (0.2 + 0.8e^t)^{100}
\]

(c) If the moment-generating function of $X$ is $M(t) = e^{5t+2t^2}$, then

\[
E(X) = 5
\]
\[
Var(X) = 4
\]

The p.d.f. of $X$ is $f(x) = \frac{1}{\sqrt{8\pi}}e^{-\frac{(x-5)^2}{8}}, -\infty < x < \infty$

(d) One of four different prizes was randomly put into each box of a cereal. If Grace decided to buy this cereal until it obtained at least one of each of the four different prizes, what is the expected number of boxes of cereal that must be purchased?

Answer : $1 + \frac{4}{3} + \frac{4}{2} + \frac{4}{1} = \frac{25}{3}$

(e) Let the r.v. $X$ has a $\chi^2$ distribution with degree of freedom 4, then

the p.d.f. of $X$ is $\frac{1}{4}xe^{-x/2}, \quad x > 0$

\[
M_X(t) = \frac{1}{(1-2t)^{2}}, \quad t < \frac{1}{2}
\]
(6pts) 4. Let $X$ have a geometric distribution with mean $\frac{1}{p}$, $0 < p < 1$, derive the moment-generating function for $X$ and compute the mean and variance of $X$ by using $E(X) = M'(0)$ and $Var(X) = M''(0) - [M'(0)]^2$.

Proof: $f(x) = qx^{x-1}p$, $x \geq 1$, $p + q = 1$.

$$M(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} q^{x-1}p = \frac{q}{p} \sum_{x=1}^{\infty} (qe^t)^x = \frac{p}{q} \frac{qe^t}{1-qe^t}$$

$$M'(0) = \frac{pe^t(1-qe^t)-pe^t(-qe^t)}{(1-qe^t)^2} \bigg|_{t=0} = \frac{p^2+pq}{p^2} = \frac{1}{p}$$

$$M''(0) = \frac{pe^t(1-qe^t)-2pe^t(-qe^t)}{(1-qe^t)^3} \bigg|_{t=0} = \frac{p^2+2pq}{p^3} = \frac{p+q}{p^3}$$

Thus, $E[X] = M'(0) = \frac{1}{p}$ and $Var(X) = M''(0) - [M'(0)]^2 = \frac{1-p}{p^3}$.

(6pts)(5) Write Matlab commands to plot each of the following probability density functions with a given random variable $X$.

(a) $X \sim \text{b}(30, 0.8)$.
(b) $Y \sim \chi^2(4)$.
(c) $W \sim \text{N}(1, 4)$.

(a) X0=0.1:0.3; X=binopdf(X0,30,0.8); bar(X0,X,0.8)
(b) X0=0.2:0.2:8; Y=chi2pdf(X0,4); plot(X0,Y,'b^-')
(c) X0=-5:0.2:7; W=normpdf(X0,1,2); plot(X0,W,'mv-')
(6pts) The p.d.f. of time $X$ to failure of an electronic component is 

$$f(x) = \frac{2x}{10^6}e^{-(x/1000)^2}, \quad 0 < x < \infty$$

(a) Computer $P(X > 2000)$.

(b) Determine the 75th percentile, $\pi_{0.75}$, of the distribution.

(c) Find the 10th and 60th percentiles, $\pi_{0.10}$, $\pi_{0.60}$.

Answer:

(a) $P(X > 2000) = e^{-4} \approx 0.0183$.

(b) $q_3 = \pi_{0.75} \approx 1177.4$.

(c) $\pi_{0.10} \approx 324.6$, $\pi_{0.60} \approx 957.2$. 
(12pts)(7) Write down the following probability density functions and derive their moment generating functions.

(a) Normal distribution with mean 3, variance 4.

\[ f(x) = \frac{1}{2\sqrt{2\pi}}e^{-(x-3)^2/8}, \quad -\infty < x < \infty, \quad \phi(t) = E[e^{tX}] = e^{3t+2t^2}. \]

Proof:

\[
\begin{align*}
\phi(t) &= \int_{-\infty}^{\infty} [e^{tx} \frac{1}{2\sqrt{2\pi}}e^{-(x-3)^2/8}] dx \\
&= \int_{-\infty}^{\infty} \left\{ \frac{1}{2\sqrt{2\pi}}e^{-(x-(3+4t))^2+(24t+16t^2)/8} \right\} dx \\
&= e^{3t+2t^2} \int_{-\infty}^{\infty} \left\{ \frac{1}{2\sqrt{2\pi}}e^{-y^2/8} \right\} dy \\
&= e^{3t+2t^2}
\end{align*}
\]

(b) \(\chi^2\) distribution with the degrees of freedom 12.

\[ f(x) = \frac{1}{\Gamma(6) \times 2^6} x^{5} e^{-x/2}, \quad x > 0, \quad \phi(t) = E[e^{tX}] = \frac{1}{(1-2t)^6}. \]

Proof:

\[
\begin{align*}
\phi(t) &= \int_{0}^{\infty} [e^{tx} \frac{1}{\Gamma(6) \times 2^6} x^{5} e^{-x/2}] dx \\
&= \frac{1}{\Gamma(6) \times 2^6} \int_{0}^{\infty} x^{5} e^{-(1-2t)x/2} dx \\
&= \frac{1}{\Gamma(6) \times 2^6} \int_{0}^{\infty} e^{-y/(1-2t)} y^{5} d\left( \frac{2y}{1-2t} \right) \\
&= \frac{1}{\Gamma(6) \times 2^6} \cdot 2^6 \frac{1}{(1-2t)^6} \int_{0}^{\infty} e^{-y} y^{6-1} dy \\
&= \frac{1}{(1-2t)^6}
\end{align*}
\]