CS5371
Theory of Computation

Lecture 9: Automata Theory VII
(Pumping Lemma, Non-CFL)

Objectives

- Introduce Pumping Lemma for CFL
- Apply Pumping Lemma to show that some languages are non-CFL

Pumping Lemma for CFL

Theorem: If L is a CFL, then

there is a number p (pumping length)

where, if w is any string in L of length at least p,

we can find u,v,x,y,z with $w = uvxyz$ and

- for each $i \geq 0$, uv^ixy^iz is in L
- $|vy| > 0$, and
- $|vxy| \leq p$

Proof of Pumping Lemma

- Let b be the maximum branching factor in the parse tree of any string in L
  - that is, the right side of any rule has at most b terminals and variables)
- We shall use $p = b^{|V|+1}$ to prove the lemma
- Observation: What is the minimum height of the parse tree for a string w with length at least p?

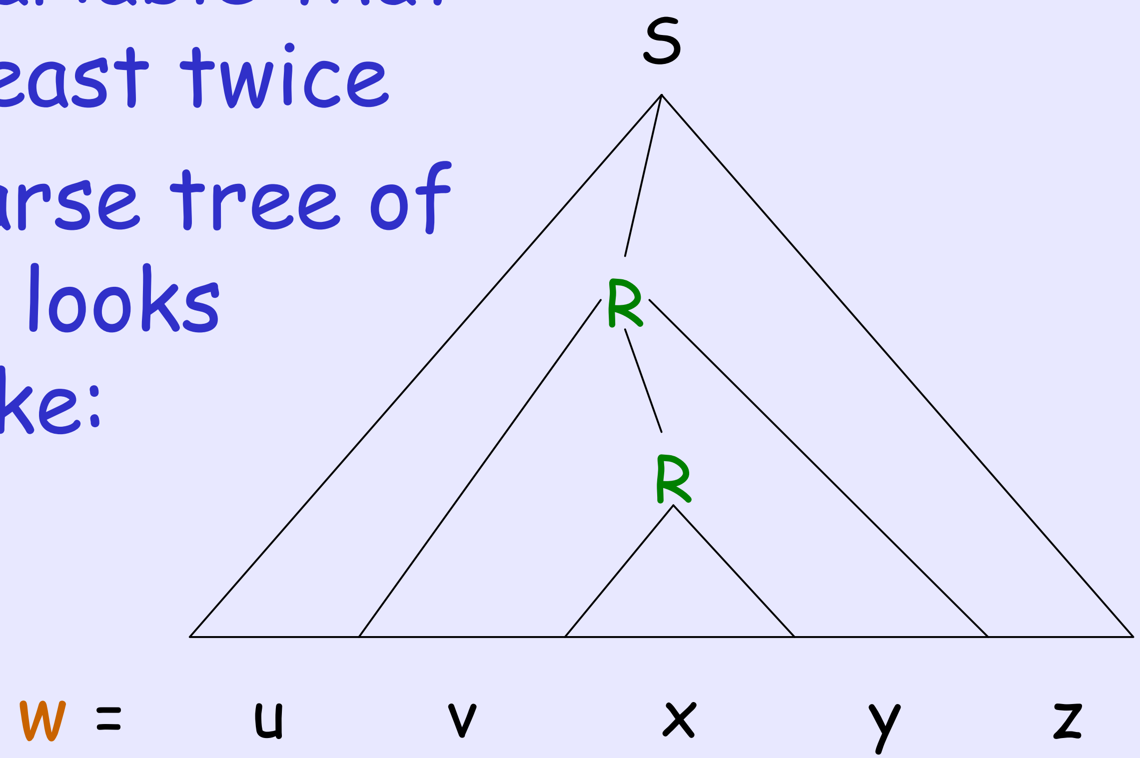
Proof of Pumping Lemma (2)

- Height of the parse tree $\geq |V| + 1$
  ➔ some path in tree $\geq |V|+2$ nodes
- Only one such node can be a terminal
  ➔ at least $|V|+1$ variable on the path
- What does that mean?

Some variable appears at least twice

Proof of Pumping Lemma (3)

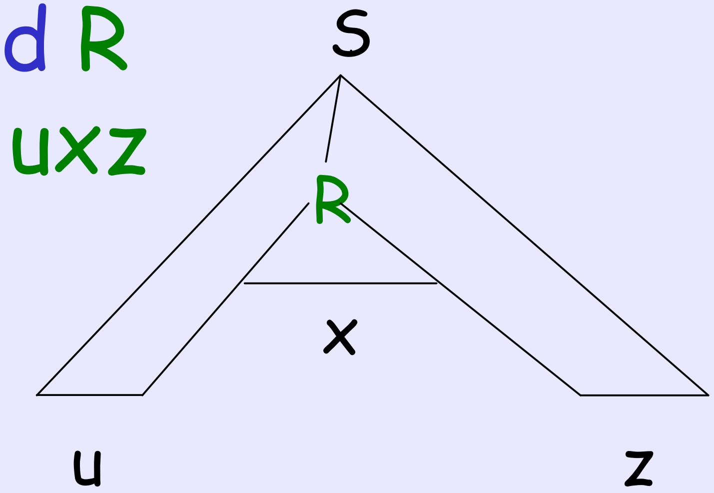
- Let R be a variable that appears at least twice
- Then, the parse tree of the string w looks something like:

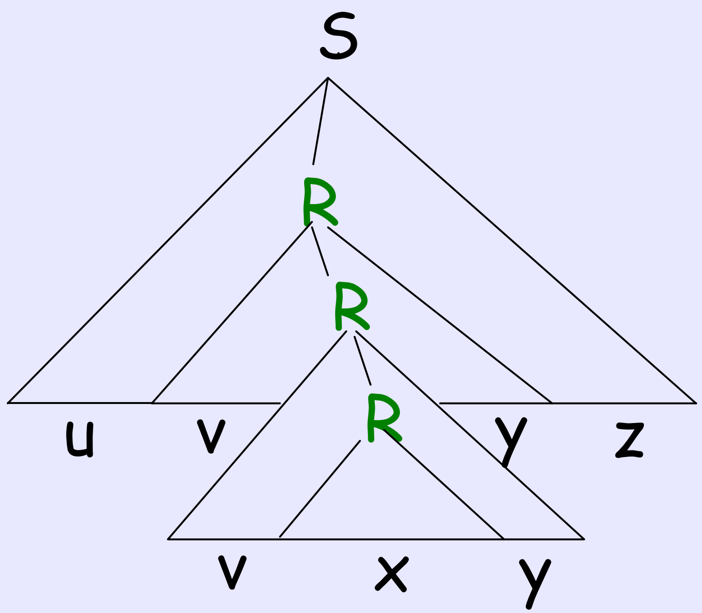


So, uv^ixy^iz is in L for any $i \geq 0$ (why??)

uv^ixy^iz is in L for any $i \geq 0$

- Facts: R derives x, R derives vxy
- Since S derives uRz, and R derives x, S can derive uxz





- Since S derives uvRyz and R derives vxy, S can derive uvvxyyz

Proof of Pumping Lemma (5)

- To complete the prove, we need to show

$$|vy| > 0 \text{ and } |vxy| \leq p$$

- The current construction cannot, but we can do so if we further restrict:

(1) parse tree is the smallest among all that can generate the string w

(2) R is chosen from the lowest |V|+1 variables in the longest root-to-leaf path

$|vy| > 0$

- Suppose on the contrary that $|vy| = 0$
  ➔ Both v and y are empty strings
- Then in the parse tree, we replace
  "Subtree of R that generates vxy"
  by "Subtree of R that generates x"
- Resulting parse tree will also generate
  w (why?), but it has fewer nodes
  ➔ contradiction occurs

$|vxy| \leq p$

- R is chosen from the lowest $|V| + 1$ variables in the longest root-to-leaf path
- Consider subtree of R that generates vxy

  Its height is at most $|V|+1$ (why?)

  ➔ It has at most $b^{|V|+1}$ leaves

  ➔ Thus, vxy has at most p characters (as $p = b^{|V|+1}$)

Recall: b = maximum branching factor

Non-CFL (example 1)

Theorem: The language

$$A = \{a^n b^n c^n \mid n \geq 0\}$$

is not a context-free language.

How to prove?

By contradiction, using pumping lemma

First thing: Assume that A is CFL

Proof (example 1)

- Let p be the pumping length
- Let $w = a^p b^p c^p$ in A, and consider partition w into any u,v,x,y,z such that $w = uvxyz$
- Two possible cases:

  Case 1: Both v and y have only one type of char

  Case 2: v or y has more than one type of char
- In both cases, $uvvxyyz$ is not in A (why?)
- Thus, we find a string at least p long in A that does not satisfy pumping lemma

  ➔ contradiction occurs

Non-CFL (example 2)

Theorem: The language

$$B = \{a^i b^j c^k \mid 0 \leq i \leq j \leq k\}$$

is not a context-free language.

How to prove?

By contradiction, using pumping lemma

First thing: Assume that B is CFL

Proof (example 2)

- Let p be the pumping length
- Let $w = a^p b^p c^p$ in B, and consider partition w into any u,v,x,y,z such that $w = uvxyz$
- Two possible cases:

  Case 1: Both v and y have only one type of char

  Case 2: v or y has more than one type of char
- We can see that for Case 2, $uvvxyyz$ cannot be in B
- How about Case 1?

Proof (example 2)

- Unfortunately, for Case 1, if v = b, y = c, then the string uvvxyyz is always in B…
- So, how to get a contradiction??

- We divide Case 1 into two subcases:

  Subcase 1.1: char a not appear in both v and y

  Subcase 1.2: char a appears in v or y

Proof (example 2)

- For Subcase 1.1 (char a not appear in v and y), uxz cannot be in B [why?]
- For Subcase 1.2 (char a appears in v or y), uvvxyyz cannot be in B [why?]
- Thus, we find a string at least p long in B that does not satisfy pumping lemma
  → contradiction occurs

Non-CFL (example 3)

Theorem: The language

$$C = \{ww \mid w \text{ in } \{0,1\}^*\}$$

is not a context-free language.

How to prove?

By contradiction, use pumping lemma on

$0^p 1^p 0^p 1^p$

Proof (example 3)

- When $w = 0^p1^p0^p1^p = uvxyz$, what can be the corresponding vxy?
  - Case 1: vxy appears in the first half
  - Case 2: vxy appears in the second half
  - Case 3: vxy includes the middle '10'
- For Cases 1 or 2, $uvvxyyz$ not in C (why?)
- For Case 3, u must start with 0^p, and z must end with 1^p (because $|vxy| \leq p$ and vxy includes the middle '10')

  ➔ Then, uxz cannot be in C (why?)

CFL is closed under all regular operations

- Union: We have seen that before
- Concatenation:

  Let G_A and G_B be CFGs for two CFLs A and B, using different sets of variables

  Let S_A and S_B be their start variables

  Combine the rules, add rule $S \rightarrow S_A\ S_B$
- Star: Add rule $S \rightarrow S\ S_A \mid \varepsilon$

CFL closed under complement?

- What is the complement of
  $A = \{a^n b^n c^n \mid n \geq 0\}$?

Assume $\Sigma = \{a, b, c\}$

- The complement of A includes:
  - strings containing ba, ca, or cb;
  - strings $a^i b^j c^k$ with $i \neq j$ or $j \neq k$

  ➔ the complement of A is a CFL (why??)

- As A is not a CFL, what can we conclude?

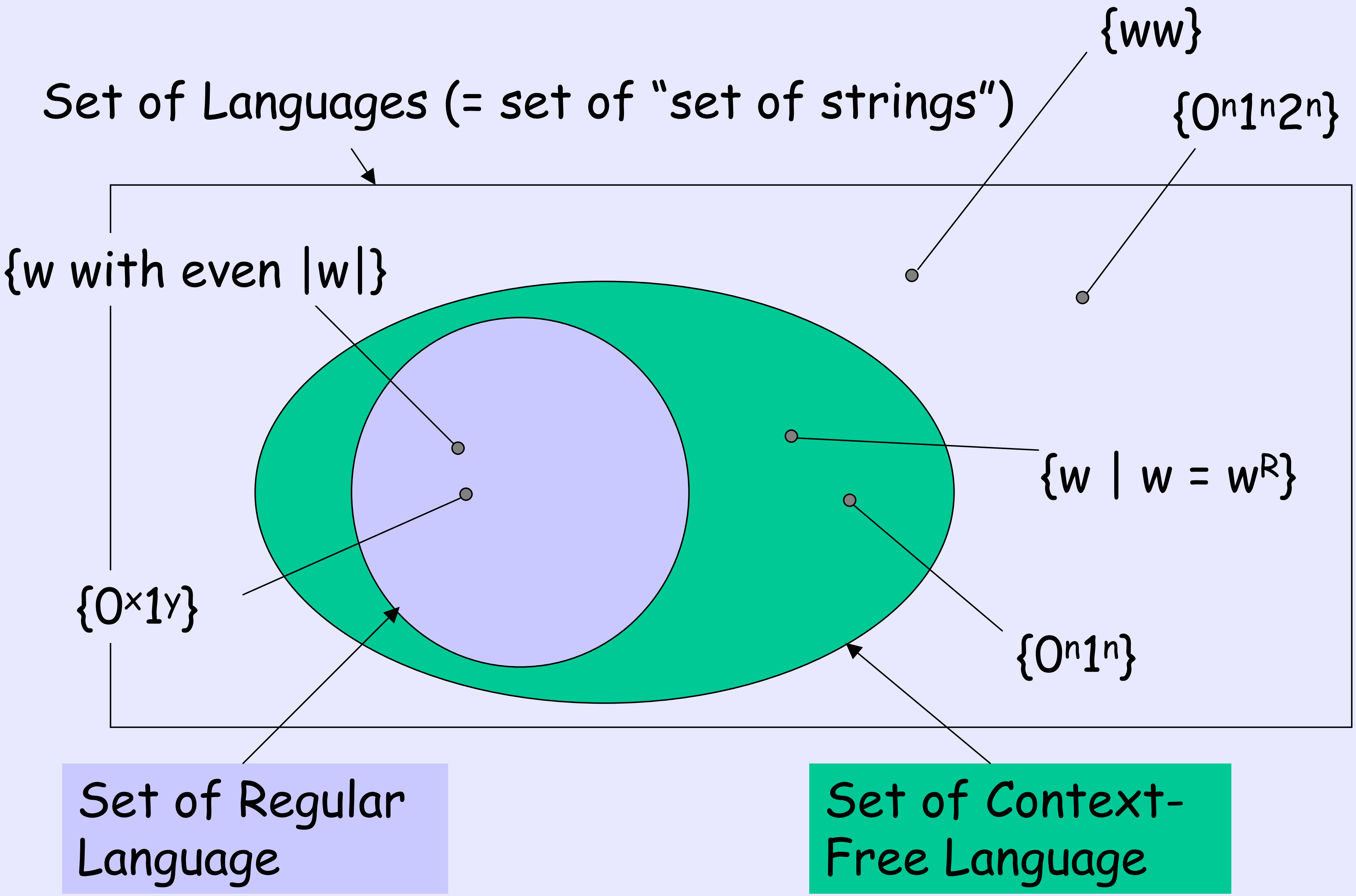
CFL closed under intersection?

- Is $A = \{a^n b^n c^m \mid n,m \geq 0\}$ a CFL?
- Is $B = \{a^m b^n c^n \mid n,m \geq 0\}$ a CFL?
- What is the intersection of A and B? Is it a CFL?
- What can we conclude?

What we have learnt so far?

- PDA = CFG
  - Prove by Construction
- Properties of CFG
  - Ambiguous, Chomsky Normal Form
- Pumping Lemma
  - Prove by Contradiction (using Parse Tree)
- Existence of non-CFL

Language Hierarchy



Next Time

- Turing Machine
  - A even more power computer