CS5371
Theory of Computation

Lecture 24: Complexity IX
(PSPACE-complete, L, NL, NL-complete)

Objectives

- PSPACE-complete languages + Examples
- The classes L and NL
- NL-complete languages + Examples

PSPACE-complete

Definition: A language B is PSPACE-complete if it satisfies the two conditions below:

1. B is in PSPACE
2. Every other language in PSPACE can be polynomial time reducible to B

If B is just satisfies Condition 2, we say B is PSPACE-hard

Question: Why don't we use polynomial space reducible?

Quantified Boolean Formula

- Mathematical statements usually involve quantifiers: $\forall$ (for all) and $\exists$ (there exists)
  - E.g., $\forall x\ F(x)$ means for every value of x, the statement F(x) is TRUE
  - E.g., $\exists x\ F(x)$ means there exists some value of x such that F(x) is TRUE
- Boolean formulas with quantifiers are called quantified Boolean formulas
  - E.g., $\exists y\ (y = x+1)$ and $\forall x\ (\exists y\ (y > x))$ are quantified Boolean formulas

Quantified Boolean Formula (2)

- The scope of a quantifier is the fragment of statement that appears within the matched parentheses following the quantified variable
  - E.g., the scope of $\exists y$ in $\exists y(y = x+1)$ is $(y = x+1)$
- If each variable in a formula appears within the scope of some quantifier, the formula is said to be fully quantified
  - A fully quantified Boolean formula is always either TRUE or FALSE

TQBF is PSPACE-complete

Let TQBF be the language

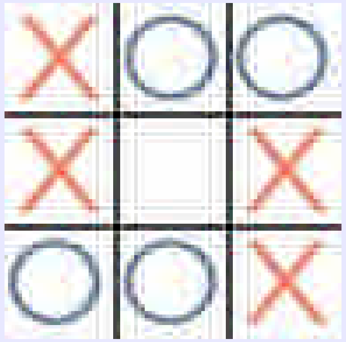
$\{\langle F \rangle \mid F$ is a true fully quantified Boolean formula $\}$

Theorem: TQBF is PSPACE-complete

PSPACE-complete [more examples]

The generalized version of some common games we play are PSPACE-complete:

Tic-Tac-Toe



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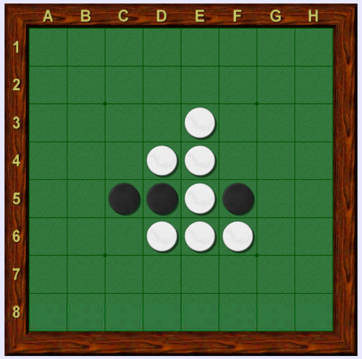

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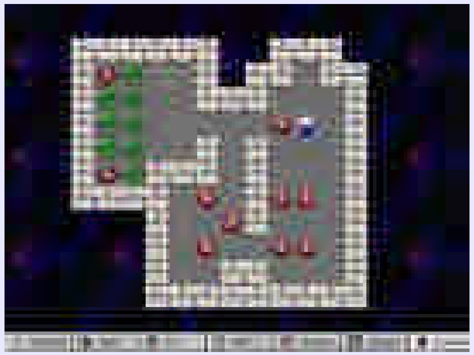

Two-Tape TM (with Read-Only tape)

- We now introduce a TM with two tapes:
  1. A read-only input tape
  2. A read/write working tape
- For this TM to operate, the input tape head always remains on the portion of the tape containing the input
- The space complexity of an algorithm is now the number of working tape cells used

The Classes L and NL

Definition:

1. L is the class of languages that are decidable in logarithmic space on a two-tape DTM. In other words,

$$L = SPACE(\log n)$$

2. NL is the class of languages that are decidable in logarithmic space on a two-tape NTM. In other words,

$$NL = NSPACE(\log n)$$

Example Language in L

Let A be the language

$$\{0^k 1^k \mid k > 0\}$$

Theorem: A is in L

Example Language in NL

Let PATH be the language

$\{ \langle G,s,t \rangle \mid G$ is a directed graph that has a directed path from s to t $\}$

Theorem: PATH is in NL

Log space Reducible

- A log space transducer is a DTM with a read-only input tape, a write-only output tape, and an $O(\log n)$-cell read/write work tape
- A log space transducer M computes a function $f: \Sigma^* \rightarrow \Sigma^*$ where $f(w)$ is the string remaining on output tape after M halts when it is started with w on its input tape
- We call f a log space computable function

Log space Reducible (2)

Definition: A language A is log space reducible to a language B, written as $A \leq_L B$, if some log space computable function f exists such that

$$w \text{ in } A \Leftrightarrow f(w) \text{ in } B$$

Properties of log space reducible

Theorem: If $A \leq_L B$ and $B \in L$, then $A \in L$.

Proof Idea: To show that "deciding whether an input w is in A" can be done in $O(\log n)$ space, a tempting approach is to perform log space reduction from A to B, obtaining $f(w)$ and then decide if $f(w)$ is in B or not…

Properties of log space reducible (2)

Problem: Unfortunately, $f(w)$ may be very long, so that the overall space usage is not in $O(\log n)$

However, observe that the decider for B, say M_B, does not need to have all $f(w)$ stored in the input tape at the same time, as long as when M_B needs to read a particular character, the character is ready for it to read → This can be done by a TM M_A that uses $O(\log n)$ space only

Properties of log space reducible (3)

... Now, to decide if w is in A, we make use of TM M_A and M_B. The total space required is $O(\log n)$ for M_A and $1+ \log |f(w)|$ for M_B.

It remains to bound the value of $|f(w)|$. Since $f(w)$ is generated from a log space transducer which halts from all inputs, $|f(w)|$ is at most the maximum number of configurations this transducer can use, which is $|w|2^{O(\log |w|)}$ ➔ space required for $M_B = 1+ \log |f(w)| = O(\log |w|) = O(\log n)$

NL-complete

Definition: A language B is NL-complete if it satisfies the two conditions below:

1. B is in NL
2. Every other language in NL can be log space reducible to B

Corollary: If any NL-complete language is in L, then L = NL

Question: Why don't we use polynomial time reducible?

PATH is NL-complete

Theorem: PATH is NL-complete.

Proof: See Chapter 8.5 (page 325)

Corollary: NL $\subseteq$ P

Proof: Any language C in NL is log space reducible to PATH. Thus, the transducer uses O(log n) space, so it runs in poly-time. Thus, C is poly-time reducible to PATH.

As PATH is in P, the proof completes.

PATH is coNL

Theorem: PATH is coNL.

Proof: See Chapter 8.6 (page 327)

Corollary: NL = coNL

Proof: We show that (1) NL $\subseteq$ coNL and (2) coNL $\subseteq$ NL. (see next slide)

NL = coNL

(1) For any $x \in NL$, we have $x \leq_L$ PATH since PATH is NL-complete. Then by the same reduction function, we have $x' \leq_L$ PATH'. Thus, $x' \in NL$ since PATH' is in NL, so $x \in coNL$. Thus, $NL \subseteq coNL$

(2) For any $x \in coNL$, $x' \in NL$. Similarly, we have $x' \leq_L$ PATH and $x \leq_L$ PATH'. Again, since PATH' is in NL, so that $x \in NL$. Thus, $coNL \subseteq NL$.

Summary

$$L \subseteq NL \subseteq P \subseteq NP \subseteq PSPACE \subseteq EXPTIME$$