CS5371
Theory of Computation

Lecture 11: Computability Theory II
(TM Variants, Church-Turing Thesis)

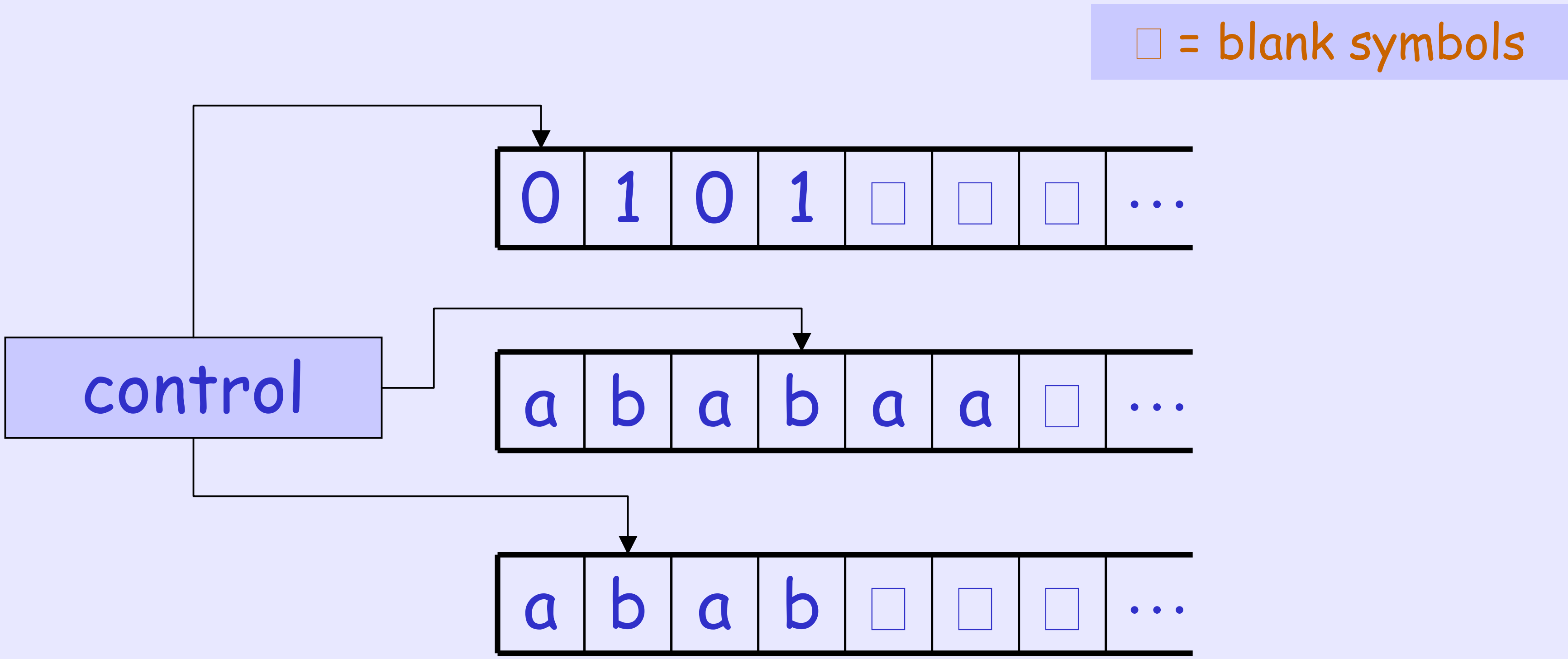
Objectives

- Variants of Turing Machine
  - With Multiple Tapes
  - With Non-deterministic Choice
  - With a Printer
- Introduce Church-Turing Thesis
  - Definition of Algorithm

Variants of TM

- Similar to the original TM
- One example: TM such that the tape head can move left, right, or stay
  - the class of languages that are recognized by this new kind of TM = the class of languages that are recognized by original TM (Why?)
- There are more variants…

Variant 1: Multi-Tape TM



It is like a TM, but with several tapes

Multi-tape TM (2)

- Initially, the input is written on the first tape, and all other tapes blank
- The transition function of a k-tape TM has the form

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{ L, R, S \}^k$$

- Obviously, given a TM, we can find a k-tape TM that recognizes the same language
- How about the converse?

Multi-tape TM = one-tape TM

Theorem: Given a k-tape TM, we can find an equivalent TM (that is, a TM that recognizes the same language).

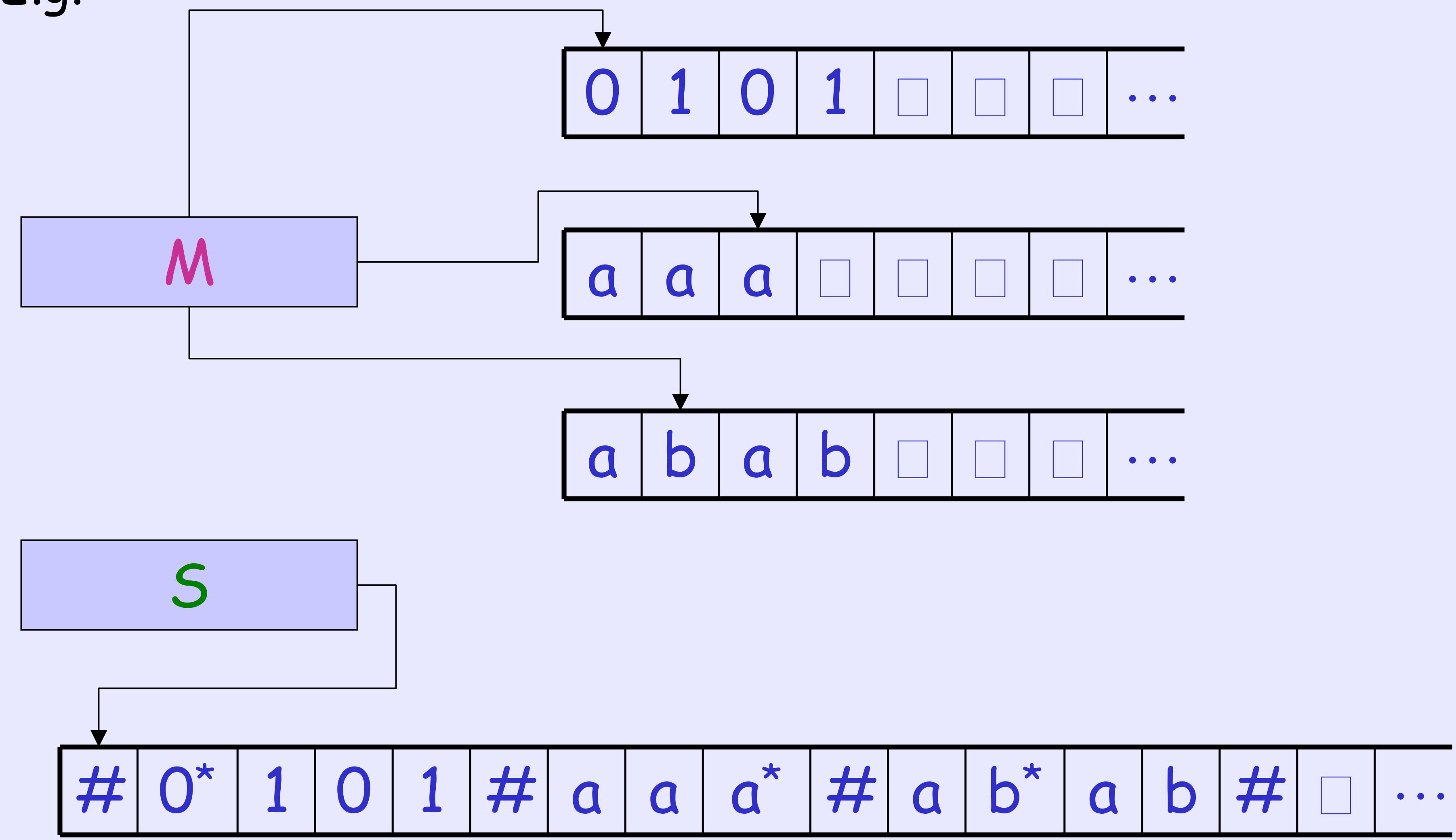
Proof: Let M be the k-tape TM (with multiple tape). We show how to convert M into some TM S (with single tape).

Multi-tape TM = one-tape TM

1. To simulate k tapes, S separates the contents of different tapes by #
2. To simulate the tape heads, S marks the symbol under each tape head with a star (The starred symbols are just new symbols in the tape alphabet of S)

→ We can now think of the tape of S to be containing k "virtual" tapes and their tape heads

E.g.



Note: $\Gamma_M = \{0,1,a,b,\square\}$ and $\Gamma_S = \{0,1,a,b,\square,\#,0^*,1^*,a^*,b^*,\square^*,\#^*\}$

The Simulation

On input $w = w_1w_2...w_n$

Step 1. S stores in the tape

$\# \ w_1^* w_2 ... w_n \ \# \ \square^* \# \ \square^* \# \ ... \ \#$

Step 2. S scans from the first # to the $(k+1)^{st}$ # to find out what symbols are under each virtual tape head

Then, S goes back to the first # and updates the virtual tapes according to what M's transition function will do

Step 3. If M accepts, accept w; If M rejects, reject w; Else, repeat Step 2

More on the Simulation

After the transition, the virtual tape head may be on top of the # symbol. Questions: (1) What do we know? (2) What should we do?

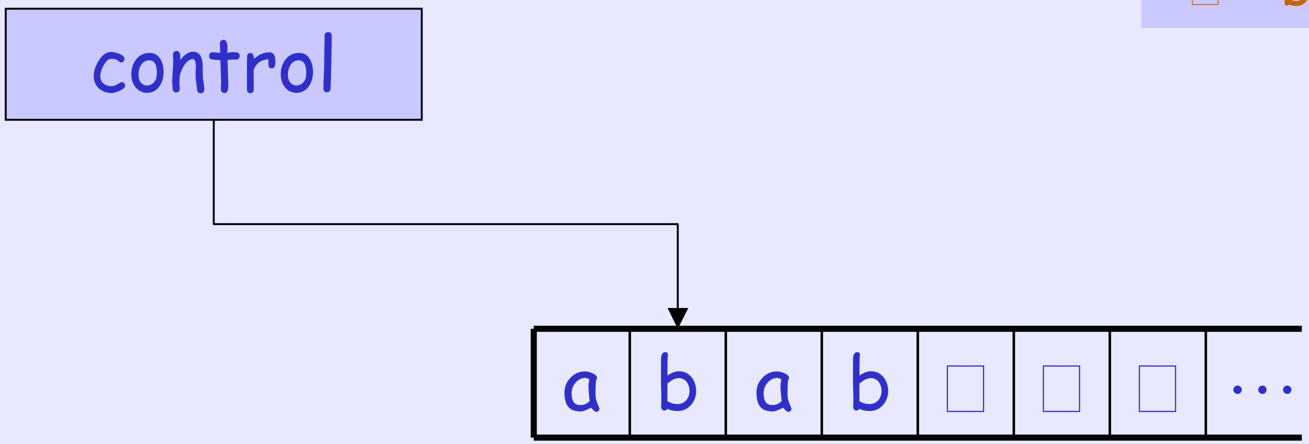
Answer:

(1) The tape head of the virtual tape has moved to the unread blank portion of the virtual tape

(2) In this case, we overwrite # by □*, shifts the tape contents of S from this cell (i.e., #) to the rightmost #, one unit to the right. After that, comes back and continues the simulation

Variant 2: NTM

□ = blank symbols



It is like a TM, but with non-deterministic control

Computation of NTM

- The transition function of NTM has the form

$$\delta : Q \times \Gamma \rightarrow 2^{Q \times \Gamma \times \{L, R\}}$$

- For an input w, we can describe all possible computations of NTM by a computation tree, where

  root = start configuration,

  children of node C = all configurations that can be yielded by C

- The NTM accepts the input w if some branch of computation (i.e., a path from root to some node) leads to the accept state

NTM = TM

Theorem: Given an NTM that recognizes a language L, we can find a TM that recognizes the same language L.

Proof: Let N be the NTM. We show how to convert N into some TM D. The idea is to simulate N by trying all possible branches of N's computation. If one branch leads to an accept state, D accepts. Otherwise, D's simulation will not terminate.

NTM = TM (Proof)

- To simulate the search, we use a 3-tape TM for D
  - first tape stores the input string
  - second tape is a working memory, and
  - third tape "encodes" which branch to search
- What is the meaning of "encode"?

NTM = TM (Proof)

- Let $b = |Q \times \Gamma \times \{ L, R \}|$, which is the maximum number of children of a node in N's computation tree.
- We encode a branch in the tree by a string over the alphabet {1,2,…,b}.
  - E.g., 231 represents the branch:

    root r → r's 2nd child c →

    c's 3rd child d → d's 1st child

NTM = TM (Proof)

On input string w,

Step 1. D stores w in Tape 1 and □ in Tape 3

Step 2. Repeat

2a. Copy Tape 1 to Tape 2

2b. Simulate N using Tape 2, with the branch of computation specified in Tape 3.

Precisely, in each step, D checks the next symbol in Tape 3 to decide which choice to make. (Special case …)

NTM = TM (Proof)

2b [Special Case].

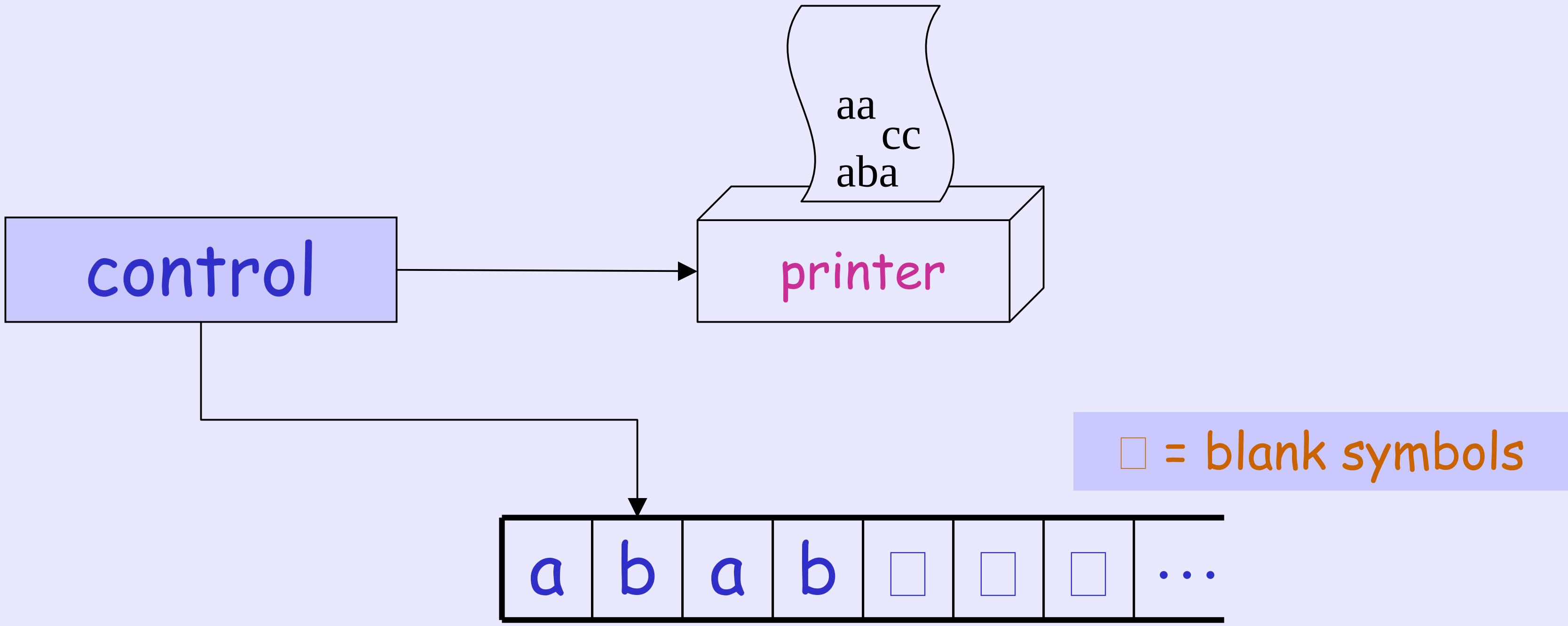
1. If this branch of N enters accept state, accepts w
2. If no more chars in Tape 3, or a choice is invalid, or if this branch of N enters reject state, D aborts this branch

2c. Copy Tape 1 to Tape 2, and update Tape 3 to store the next branch (in Breadth-First Search order)

NTM = TM (Proof)

- In the simulation, D will first examine the branch ε (i.e., root only), then the branch 1 (i.e., root and 1st child only), then the branch 2, and then 3, 4, …, b, then the branches 11, 12, 13, …, 1b, then 21, 22, 23, …, 2b, and so on, until the examined branch of N enters an accept state (what if N enters a reject state?)
- If N does not accept w, the simulation of D will run forever
- Note that we cannot use DFS (depth-first search) instead of BFS (why?)

Variant 3: Enumerator



It is like a TM, but with a printer

Enumerator (2)

- An enumerator E starts with a blank input tape
- Whenever the TM wants to print something, it sends the string to the printer
- If the enumerator does not halt, it may print an infinite list of strings
- The language of E = the set of strings that are (eventually) printed by E
  - Note: E may generate strings in any order, and with repetitions

Enumerator (3)

Theorem: Let L be a language.

(1) If L is enumerated by some enumerator, there is a TM that recognizes L.

(2) If L is recognized by some TM, there is an enumerator that enumerates L.

Enumerator (4)

Proof of (1): Let E be the enumerator that enumerates L. Consider the following TM M:

M = On input w:

Step 1. Run E. Whenever E wants to print, compare the string with w.

If they are the same, accept w.

Otherwise, continue to run E.

Thus, M accepts exactly the strings in L

→ there is a TM that recognizes L

Enumerator (5)

Proof of (2): Let M be the TM that recognizes L. Consider the following enumerator E:

E = On input x:

Step 1. Repeat for i = 1,2,3,... (forever)

1a. Run M for i steps on the first i strings in Σ^* (sorted by length, then lex order) E.g., when $\Sigma = \{0,1\}$, the order of strings is: ε, 0, 1, 00, 01, 10, ...

1b. If M accepts a string w, print w

Enumerator (6)

- In the Proof of (2), we see that if a string is accepted by M, it will be printed by E eventually (why?), though it will be printed infinitely many times (why?)

- Recall that Turing-recognizable language is also called recursively enumerable language. The latter term actually originates from enumerator

Hilbert's 10th Problem

- In 1900, David Hilbert delivered a famous talk in International Congress of Mathematicians
- He identified 23 math problems which he thinks are important in the coming century
- The 10th Problem: Given a multi-variable polynomial F with integral coefficients (such as $F(x,y,z) = 6x^3yz^2 + 3xy^2 - 27$).

  Any algorithm can tell if we have an integral root for F = 0? [E.g., in this case, x=y=1, z=2 is an integral root for F(x,y,z)=0]

Hilbert's 10^{th} Problem (2)

- However, what is meant by an algorithm?
- Roughly speaking, one meaning of algorithm is: a set of steps for solving a problem, such that when we are provided with unlimited supply of pencils and papers, we can blindly follow these steps and solve the problem
- There is no precise definition, until in 1936, two separate papers, one from Alonzo Church and one from Alan Turing, try to define it

Church-Turing Thesis

- Turing restricts that each algorithm step must be simple enough for a TM to perform
- Church's definition of algorithm is based on something called λ-calculus
- Surprisingly, these two definitions are shown to be equivalent!! (That is, a problem P can be solved by an algorithm with Turing's definition if and only if P can be solved by some algorithm with Church's definition)
  - Later (in 1970), Yuri Matijasevič proves that, under their definition, no algorithm can test whether a multi-variable polynomial has integral root

Church-Turing Thesis (2)

- Also, it seems that all known problems that are solvable by an "algorithm" (with our "intuitive" and "non-precise" definition) are exactly the problems solvable by TM
- Therefore, Steven Kleene (1943) makes the following conjecture in his paper, which is now known as the Church-Turing Thesis:

  "If a problem is intuitively solvable, it can be solved by TM"

Solving Problem by TM (example)

- Let A be the language

  $\{ \langle G \rangle \mid G = \text{undirected connected graph}\}$

  where $\langle G \rangle$ the encoding of G
- That is, given an undirected graph G, we want to determine if G is connected
- How to solve it by TM?

Solving Problem by TM (example)

M = "On input $\langle G \rangle$

Step 1. Select first node of G and mark it

Step 2. Repeat the following stage until no new nodes are marked:

2a. For each node in G, mark it if it is attached to a marked node

Step 3. Scan all nodes. If all are marked, accept. Otherwise, reject.

Next Time

- Decidable Language
  - Can be decided by some algorithm
- Undecidable Language
  - No algorithm can decide it