CS5371
Theory of Computation

Lecture 1: Mathematics Review I
(Basic Terminology)

Objectives

- Unlike other CS courses, this course is a MATH course…
- We will look at a lot of definitions, theorems and proofs
- This lecture: reviews basic math notation and terminology
  - Set, Sequence, Function, Graph, String…
- Also, common proof techniques
  - By construction, induction, contradiction

Set

- A set is a group of items
- One way to describe a set: list every item in the group inside { }
  - E.g., { 12, 24, 5 } is a set with three items
- When the items in the set has trend: use …
  - E.g., { 1, 2, 3, 4, … } means the set of natural numbers
- Or, state the rule
  - E.g., $\{ n \mid n = m^2$ for some positive integer $m \}$ means the set { 1, 4, 9, 16, 25, … }
- A set with no items is an empty set denoted by { } or $\emptyset$

Set

- The order of describing a set does not matter
  - $\{ 12, 24, 5 \} = \{ 5, 24, 12 \}$
- Repetition of items does not matter too
  - $\{ 5, 5, 5, 1 \} = \{ 1, 5 \}$
- Membership symbol $\in$
  - $5 \in \{ 12, 24, 5 \}$ $\quad$ $7 \notin \{ 12, 24, 5 \}$

Set (Quick Quiz)

- How many items are in each of the following set?
  - { 3, 4, 5, …, 10 }
  - { 2, 3, 3, 4, 4, 2, 1 }
  - { 2, {2}, {{1,2,3,4,5,6}} }
  - $\emptyset$
  - $\{\emptyset\}$

Set

Given two sets A and B

- we say $A \subseteq$ B (read as A is a subset of B) if every item in A also appears in B
  - E.g., A = the set of primes, B = the set of integers
- we say $A \subsetneq$ B (read as A is a proper subset of B) if $A \subseteq$ B but $A \neq$ B

Warning: Don't be confused with $\in$ and $\subseteq$

- Let $A = \{ 1, 2, 3 \}$. Is $\emptyset \in A$? Is $\emptyset \subseteq A$?

Union, Intersection, Complement

Given two sets A and B

- $A \cup B$ (read as the union of A and B) is the set obtained by combining all elements of A and B in a single set
  - E.g., $A = \{ 1, 2, 4 \}$ $B = \{ 2, 5 \}$
    $A \cup B = \{ 1, 2, 4, 5 \}$
- $A \cap B$ (read as the intersection of A and B) is the set of common items of A and B
  - In the above example, $A \cap B = \{ 2 \}$
- $\overline{A}$ (read as the complement of A) is the set of items under consideration not in A

Set

- The power set of A is the set of all subsets of A, denoted by 2^A
  - E.g., $A = \{ 0, 1 \}$

    $2^A = \{ \{\}, \{0\}, \{1\}, \{0,1\} \}$
  - How many items in the above power set of A?
- If A has n items, how many items does its power set contain? Why?

Sequence

- A sequence of items is a list of these items in some order
- One way to describe a sequence: list the items inside ( )
  - (5, 12, 24)
- Order of items inside ( ) matters
  - (5, 12, 24) $\neq$ (12, 5, 24)
- Repetition also matters
  - (5, 12, 24) $\neq$ (5, 12, 12, 24)
- Finite sequences are also called tuples
  - (5, 12, 24) is a 3-tuple
  - (5, 12, 12, 24) is a 4-tuple

Sequence

Given two sets A and B

- The Cartesian product of A and B, denoted by $A \times B$, is the set of all possible 2-tuples with the first item from A and the second item from B
  - E.g., $A = \{1, 2\}$ and $B = \{x, y, z\}$

  $A \times B = \{ (1,x), (1,y), (1,z), (2,x), (2,y), (2,z) \}$

- The Cartesian product of k sets, $A_1, A_2, \ldots, A_k$, denoted by $A_1 \times A_2 \times \cdots \times A_k$, is the set of all possible k-tuples with the i^{th} item from A_i

Functions

- A function takes an input and produces an output
- If f is a function, which gives an output b when input is a, we write

$$f(a) = b$$

- For a particular function f, the set of all possible input is called f's domain
- The outputs of a function come from a set called f's range

Functions

- To describe the property of a function that it has domain D and range R, we write

$$f : D \rightarrow R$$

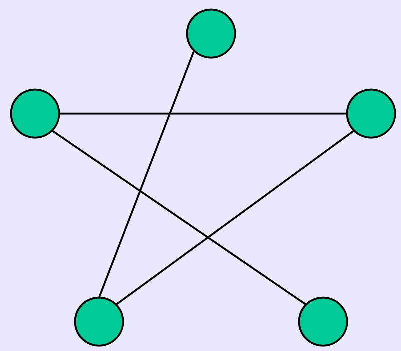
- E.g., The function add (to add two numbers) will have an input of two integers, and output of an integer
  - We write: add: $Z \times Z \rightarrow Z$

Functions (Quick Quiz)

- Guess: What does the following DOW function do?
  - DOW(9,11) = 2
  - DOW(9,12) = 3
  - DOW(9,13) = 4
  - DOW(9,17) = 1
  - DOW(10,1) = 1
- What are the domain and the range of DOW?

Graphs

- A graph is a set of points with lines connecting some of the points
- Points are called vertices, lines are called edges
- E.g.,

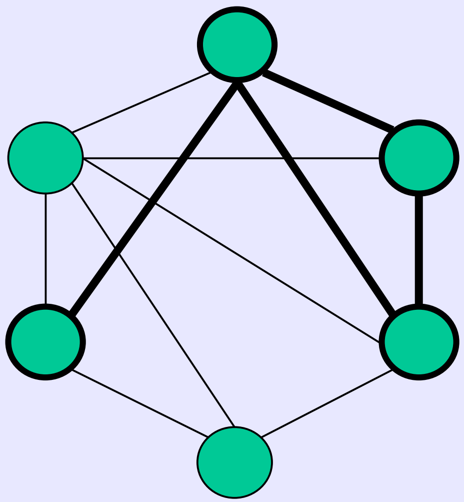


Graphs

- The number of edges at a particular vertex is the degree of the vertex
- In the previous example, 3 vertices have degree = 2
- A graph can be described by telling what are its vertices, and what are its edges. Formally, a graph G can be written as G = (V, E), where V is the set of vertices, and E is the set of edges

Graphs

- We say a graph G is a subgraph of H if vertices of G are a subset of the vertices of H, and all edges in G are the edges of H on the corresponding vertices



Subgraph G
shown darker

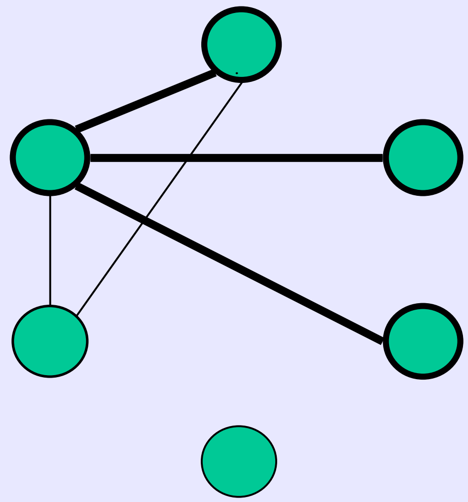
Graph H

Graphs

- A path is a sequence of vertices connected by edges
- If every two nodes have a path between them, the graph is connected
- A cycle is a path that starts and ends at the same vertex
- A tree is a connected graph with no cycles

Graphs (Quick Quiz)

- Is the following graph connected?
- Is it a tree?
- Are there any cycles?
- How about the darker subgraph?

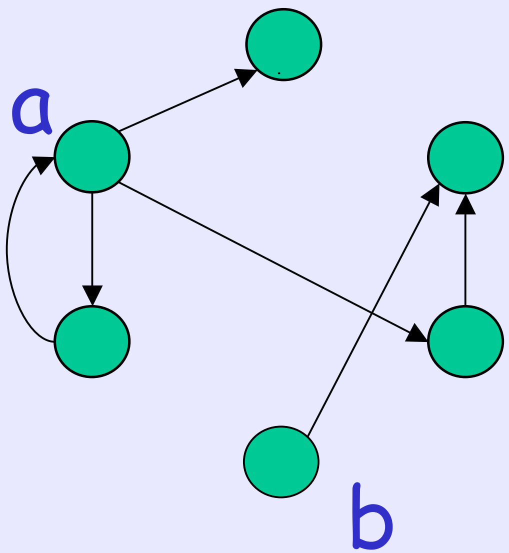


Directed Graphs

- If lines are replaced by arrows, the graph becomes directed
- The number of arrows pointing into a vertex is called in-degree of the vertex
- The number of arrows pointing from a vertex is called out-degree of the vertex
- A directed path is a path from one vertex to the other vertex, following the direction of the "arrows"

Directed Graphs

- Is there a directed path from a to b?



Strings

- An alphabet = a set of characters
  - E.g., The English Alphabet = {A,B,C,…,Z}
- A string = a sequence of characters
- A string *over* an alphabet Σ
  - A sequence of characters, with each character coming from Σ
- The length of a string w, denoted by $|w|$, is the number of characters in w
- The empty string (written as ε) is a string of length 0

Strings

Let $w = w_1w_2 \ldots w_n$ be a string of length n

- A substring of w is a consecutive subsequence of w (that is, $w_iw_{i+1} \ldots w_j$ for some $i \leq j$)
- The reverse of w, denoted by w^R, is the string $w_n \ldots w_2 w_1$
- A set of strings is called a language

Next time

- Common Proof Techniques
- Part I: Automata Theory