CS5371 Theory of Computation

Homework 4 (Suggested Solution)

1. **Ans.** Suppose on the contrary that T is decidable. Let R be its decider. Then, the following TM Q is a decider for A_{TM}:

   $Q =$ "On input $\langle M, w\rangle$,
   1. Construct a TM M' as follows:
      $M' =$ "On input x,
      1. If $x \neq \texttt{011}$, *accept*.
      2. Run M on w.
      3. If M accepts w, *accept*."
   2. Run R to decide if $\langle M'\rangle$ is in T.
   3. If yes (i.e., R accepts), *accept*.
   4. Else, *reject*."

   It is easy to check that Q runs in finite steps. Also, in Step 1, M' has the property that:

   (i) If M accepts w, $L(M') = \Sigma^*$, so that $\langle M'\rangle \in T$.
   (ii) Else, $L(M') = \Sigma^* - \{\texttt{011}\}$, so that $\langle M'\rangle \notin T$.

   So, if Q accepts $\langle M, w\rangle$, it must mean that R accepts $\langle M'\rangle$, which implies that $\langle M'\rangle \in T$, which implies M accepts w. On the other hand, if Q rejects $\langle M, w\rangle$, R rejects $\langle M'\rangle$, which in turn implies that M does not accept w.

   Thus, Q is a decider for A_{TM}, and a contradiction occurs. So, we conclude that T is undecidable.

2. In the *silly Post Correspondence Problem*, we see that if a set of dominoes S is in *SPCP* if and only if S contains a piece whose top string matches exactly the bottom string. Thus, we can easily design a TM that uses finite steps to check such a piece exists. So, *SPCP* is decidable.

3. ($\Rightarrow$) If $A \leq_m A_{TM}$, then A is Turing-recognizable because A_{TM} is Turing recognizable.

   ($\Leftarrow$) If A is Turing-recognizable, then there exists some TM R that recognizes A. That is, R would receive an input w and accept if w is in A (otherwise R does not accept). To show that $A \leq_m A_{TM}$, we design a TM that does the following: On input w, writes $\langle R, w\rangle$ on the tape and halts. It is easy to check that $\langle R, w\rangle$ is in A_{TM} if and only if w is in A. Thus, we get a mapping reduction of A to A_{TM}.

4. ($\Rightarrow$) If $A \leq_m 0^*1^*$, then A is decidable because 0^*1^* is a decidable language.

   ($\Leftarrow$) If A is decidable, then there exists some TM R that decides A. That is, R would receive an input w and accept if w is in A, reject if w is not in A. To show $A \leq_m 0^*1^*$, we design a TM Q that does the following: On input w, runs R on w. If R accepts, outputs $\texttt{01}$; otherwise, outputs $\texttt{10}$. It is easy to check that:

   $$w \in A \Leftrightarrow \text{ output of } Q \in 0^*1^*.$$

   Thus, we obtain a mapping reduction of A to 0^*1^*.

5. Let $J = \{w \mid \text{either } w = \texttt{0}x \text{ for some } x \in A_{TM}, \text{ or } w = \texttt{1}y \text{ for some } y \notin A_{TM}\}$.

- We first show that $A_{TM} \leq_m J$. To do so, we design the following TM Q: On input $\langle M, w\rangle$, write `0` followed by $\langle M, w\rangle$ in the tape and halts. It is easy to check that:

  $$\langle M, w\rangle \in A_{TM} \Leftrightarrow \text{ output of } Q \ \in J.$$

  Thus, we obtain a mapping reduction of A_{TM} to J.

- We next show that $A_{TM} \leq_m \bar{J}$. To do so, we design the following TM R: On input $\langle M, w\rangle$, write `1` followed by $\langle M, w\rangle$ in the tape and halts. It is easy to check that:

  $$\langle M, w\rangle \in \overline{A_{TM}} \Leftrightarrow \text{ output of } R \ \in J.$$

  Equivalently, we have:

  $$\langle M, w\rangle \in A_{TM} \Leftrightarrow \text{ output of } R \ \in \bar{J}.$$

  Thus, we obtain a mapping reduction of A_{TM} to $\bar{J}$.

- Since $A_{TM} \leq_m J$, we have $\overline{A_{TM}} \leq_m \bar{J}$. This shows that $\bar{J}$ is non-Turing-recognizable because $\bar{A}_{TM}$ is non-Turing-recognizable.

  Similarly, since $A_{TM} \leq_m \bar{J}$, we have $\overline{A_{TM}} \leq_m J$. So, this shows that J is non-Turing-recognizable.