CS5314 Randomized Algorithms

Lecture 11: Chernoff Bounds (Application)

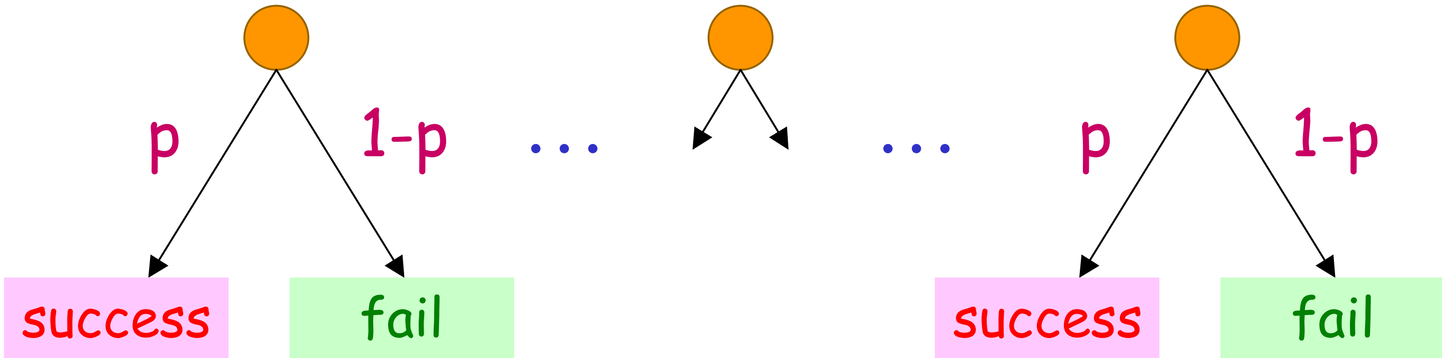
Objectives

- Apply Chernoff bounds to bound the tail and the head distributions of sum of Poisson trials
  - General case of Binomial random variable
- Revisit the Coin Flip example

Sum of Bernoulli Trials

Recall that: X = Bin(n,p) is a random variable for the sum of n independent indicators, each with success probability p

➔ Each indicator is called a Bernoulli trial

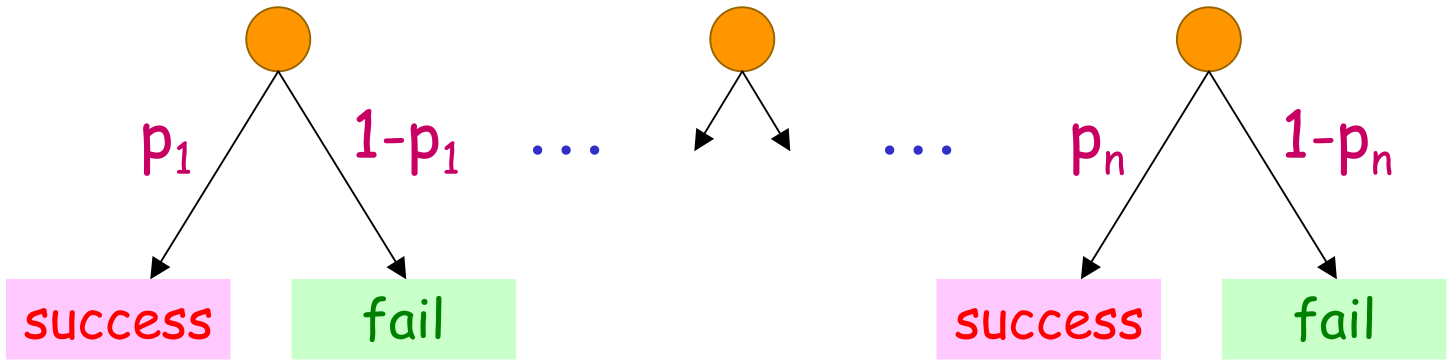


n = # trials X = total # success

Sum of Poisson Trials

Suppose that we allow each of these indicators to choose its own success probability (instead of a fixed p)

➔ Each indicator is called a Poisson trial



n = # trials   X = total # success

Bounding Sum of Poisson Trials

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent Poisson trials, such that

$$\Pr(X_i = 1) = p_i$$

Let $X = X_1 + X_2 + \ldots + X_n$

Then,

$$\mu = E[X] = E[X_1 + X_2 + \ldots + X_n]$$
$$= p_1 + p_2 + \ldots + p_n$$

Bounding Sum of Poisson Trials

We shall use Chernoff bound to bound the tail distribution $\Pr(X \geq (1+\delta)\mu)$:

Let M_X be the MGF of X, and
M_{X_i} be the MGF of X_i

Since X_i's are independent, we have

$$M_X(t) = \prod_i M_{X_i}(t) \qquad \text{... (why?)}$$

Bounding Sum of Poisson Trials

Question: What is $M_{X_i}(t)$?

By definition,

$$M_{X_i}(t) = E[e^{tX_i}]$$
$$= p_i\, e^{t(1)} + (1-p_i)\, e^{t(0)}$$
$$= p_i\, e^t + (1-p_i)$$
$$= 1 + p_i\,(e^t-1)$$
$$\leq e^{p_i (e^t-1)} \qquad \text{... (why?)}$$

Bounding Sum of Poisson Trials

So,

$$M_X(t) = \prod_i M_{X_i}(t)$$

$$\leq \prod_i e^{p_i (e^t - 1)} \qquad \text{... (why?)}$$

$$= \exp\{ \sum_i p_i(e^t - 1) \}$$

$$= e^{\mu(e^t - 1)}$$

Bounding Sum of Poisson Trials

Theorem: Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$.

Let $X = X_1+X_2+ \ldots +X_n$ and $\mu = E[X]$.

Then, for all $\delta > 0$,

$$\Pr(X \geq (1+\delta)\mu) \leq \left(e^{\delta}/(1+\delta)^{(1+\delta)}\right)^{\mu}$$

How to prove?

Proof: For any $t > 0$, we have

$$\Pr(X \geq (1+\delta)\mu)$$

$$= \Pr(e^{tX} \geq e^{t(1+\delta)\mu})$$

$$\leq E[e^{tX}] \,/\, e^{t(1+\delta)\mu}$$ ... (by which inequality?)

$$= M_X(t) \,/\, e^{t(1+\delta)\mu}$$

$$\leq e^{\mu(e^t-1)} \,/\, e^{t(1+\delta)\mu}$$

To get the best bound for $\Pr(X \geq (1+\delta)\mu)$, we now choose t so as to minimize the term

$$e^{\mu(e^t-1)} \,/\, e^{t(1+\delta)\mu}$$

Proof (cont)

Question: Which t should we choose?

Observation:

To minimize $e^{\mu(e^t-1)} / e^{t(1+\delta)\mu}$

$\Leftrightarrow$ to minimize $\log_e (e^{\mu(e^t-1)} / e^{t(1+\delta)\mu})$

So, we want to choose t so as to minimize the term $\log_e (e^{\mu(e^t-1)} / e^{t(1+\delta)\mu})$, which is

$$\mu(e^t-1) - t(1+\delta)\mu$$

By calculus, the best t will be $\log_e (1+\delta)$

Proof (cont)

So, by substituting $t = \log_e (1+\delta)$ in the previous inequality:

$$\Pr(X \geq (1+\delta)\mu) \leq e^{\mu(e^t - 1)} / e^{t(1+\delta)\mu}$$

we get:

$$\Pr(X \geq (1+\delta)\mu) \leq e^{\mu(\delta)} / (1+\delta)^{(1+\delta)\mu}$$

$$= \left(e^{\delta}/(1+\delta)^{(1+\delta)}\right)^{\mu}$$

Two Weaker but Easier Bounds

Theorem: Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$.

Let $X = X_1+X_2+ \ldots +X_n$ and $\mu = E[X]$. Then,

(1) for all $0 < \delta \le 1$,

$$\Pr(X \ge (1+\delta)\mu) \le e^{-\mu\delta^2/3}$$

(2) For $R \ge 6\mu$,

$$\Pr(X \ge R) \le 2^{-R}$$

How to prove?

Proof of (1)

To prove (1), it is sufficient to show
whenever $0 < \delta \le 1$,

$$e^{\delta}/(1+\delta)^{(1+\delta)} \le e^{-\delta^2/3}$$

Equivalently, we want to show

$$\delta - (1+\delta)\log_e(1+\delta) \le -\delta^2/3$$

(this is obtained by taking log on both sides)

Let $f(\delta) = \delta - (1+\delta)\log_e(1+\delta) + \delta^2/3$

Target: to show $f(\delta) \le 0$

Proof of (1)

Then, $f'(\delta) = 1 - (1+\delta)/(1+\delta) - \log_e(1+\delta) + 2\delta/3$

$= - \log_e(1+\delta) + 2\delta/3$

and

$$f''(\delta) = -1/(1+\delta) + 2/3$$

So, we see that

$f''(\delta) < 0$ for $0 \leq \delta < 1/2$

$f''(\delta) \geq 0$ for $1/2 \leq \delta \leq 1$

Proof of (1)

This implies that

$f'(\delta)$ is decreasing for $0 \leq \delta < 1/2$

$f'(\delta)$ is increasing for $1/2 \leq \delta \leq 1$

Next, from $f'(\delta) = -\log_e(1+\delta) + 2\delta/3$,

we see that $f'(0) = 0$ and $f'(1) < 0$

Together with the above, we conclude that

$f'(\delta) \leq 0$ in the interval $[0,1]$

Proof of (1)

Since $f'(\delta) \leq 0$ in the interval [0,1],

f is decreasing in the interval [0,1]

➔ $f(0)$ is the maximum point of f in the interval [0,1]

Thus, for $0 < \delta \leq 1$,

$$f(\delta) = \delta - (1+\delta) \log_e (1+\delta) + \delta^2/3$$

$$\leq f(0) = 0$$

This completes the proof of (1)

Proof of (2)

For (2), we want to show for $R \geq 6\mu$,

$$\Pr(X \geq R) \leq 2^{-R}$$

Let $R = (1+\delta)\mu$, so that $\delta \geq 5 > 0$

Then, we have

$$\Pr(X \geq (1+\delta)\mu) \leq (e^{\delta}/(1+\delta)^{(1+\delta)})^{\mu}$$

$$\leq (e^{(1+\delta)}/(1+\delta)^{(1+\delta)})^{\mu}$$

$$= (e/(1+\delta))^{\mu(1+\delta)}$$

$$\leq (e/6)^{R} \leq 2^{-R}$$

Bounding the Head Distribution

Theorem: Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$.

Let $X = X_1+X_2+ \ldots +X_n$ and $\mu = E[X]$.

Then, for $0 < \delta < 1$,

(1) $\Pr(X \leq (1-\delta)\mu) \leq (e^{-\delta}/(1-\delta)^{(1-\delta)})^{\mu}$

(2) $\Pr(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/2}$

How to prove?

Proof of (1): For any $t < 0$, we have

$$\Pr(X \le (1-\delta)\mu)$$
$$= \Pr(e^{tX} \ge e^{t(1-\delta)\mu})$$
$$\le E[e^{tX}] \,/\, e^{t(1-\delta)\mu}$$ ... (by which inequality?)
$$= M_X(t) \,/\, e^{t(1-\delta)\mu}$$
$$\le e^{\mu(e^t-1)} \,/\, e^{t(1-\delta)\mu}$$

To get the best bound for $\Pr(X \le (1-\delta)\mu)$, we now choose t so as to minimize the term

$$e^{\mu(e^t-1)} \,/\, e^{t(1-\delta)\mu}$$

Proof of (1)

By calculus, the best t is equal to $\log_e (1-\delta)$

So, by substituting $t = \log_e (1-\delta)$ in the previous inequality, we get

$$\Pr(X \leq (1-\delta)\mu) \leq e^{\mu(-\delta)} / (1-\delta)^{(1-\delta)\mu}$$

$$= \left(e^{-\delta}/(1-\delta)^{(1-\delta)}\right)^{\mu}$$

Proof of (2)

To prove (2), it is sufficient to show whenever $0 < \delta < 1$,

$$e^{-\delta}/(1-\delta)^{(1-\delta)} \leq e^{-\delta^2/2}$$

Equivalently, we want to show

$$-\delta - (1-\delta)\log_e(1-\delta) \leq -\delta^2/2$$

(this is obtained by taking log on both sides)

Let $g(\delta) = -\delta - (1-\delta)\log_e(1-\delta) + \delta^2/2$

Target: to show $g(\delta) \leq 0$

Proof of (2)

Then,

$$g'(\delta) = -1 + (1-\delta)/(1-\delta) + \log_e(1-\delta) + 2\delta/2$$

$$= \log_e(1-\delta) + \delta$$

and

$$g''(\delta) = -1/(1-\delta) + 1$$

So, we see that

$$g''(\delta) < 0 \qquad \text{for } 0 \leq \delta < 1$$

Proof of (2)

This implies that

$g'(\delta)$ is decreasing   for $0 \le \delta < 1$

Next, from $g'(\delta) = -\log_e(1+\delta) + 2\delta/3$,

we see that $g'(0) = 0$

Together with the above, we conclude that

$g'(\delta) \le 0$ in the interval $[0,1)$

Proof of (2)

Since $g'(\delta) \leq 0$ in the interval [0,1),
g is decreasing in the interval [0,1)
➔ $g(0)$ is the maximum point of g in the interval [0,1)

Thus, for $0 < \delta < 1$,

$$g(\delta) = - \delta - (1-\delta) \log_e (1-\delta) + \delta^2/2$$

$$\leq g(0) = 0$$

This completes the proof of (2)

Useful Corollary

Corollary: Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i = 1) = p_i$.

Let $X = X_1+X_2+ \ldots +X_n$ and $\mu = E[X]$. Then, for all $0 < \delta < 1$,

$$\Pr(|X - \mu| \geq \delta\mu) \leq 2e^{-\mu\delta^2/3}$$

How to prove?

Example: Coin Flip

Let X = # heads in n fair coin flips

By Markov: $\Pr(X \geq 3n/4) \leq 2/3$

By Chebyshev: $\Pr(X \geq 3n/4) \leq 4/n$

By Chernoff:

$$\Pr(X \geq 3n/4) = \Pr(X \geq (1.5)\mu)$$

$$\leq e^{-\mu(0.5)^2/3} = e^{-n/24}$$

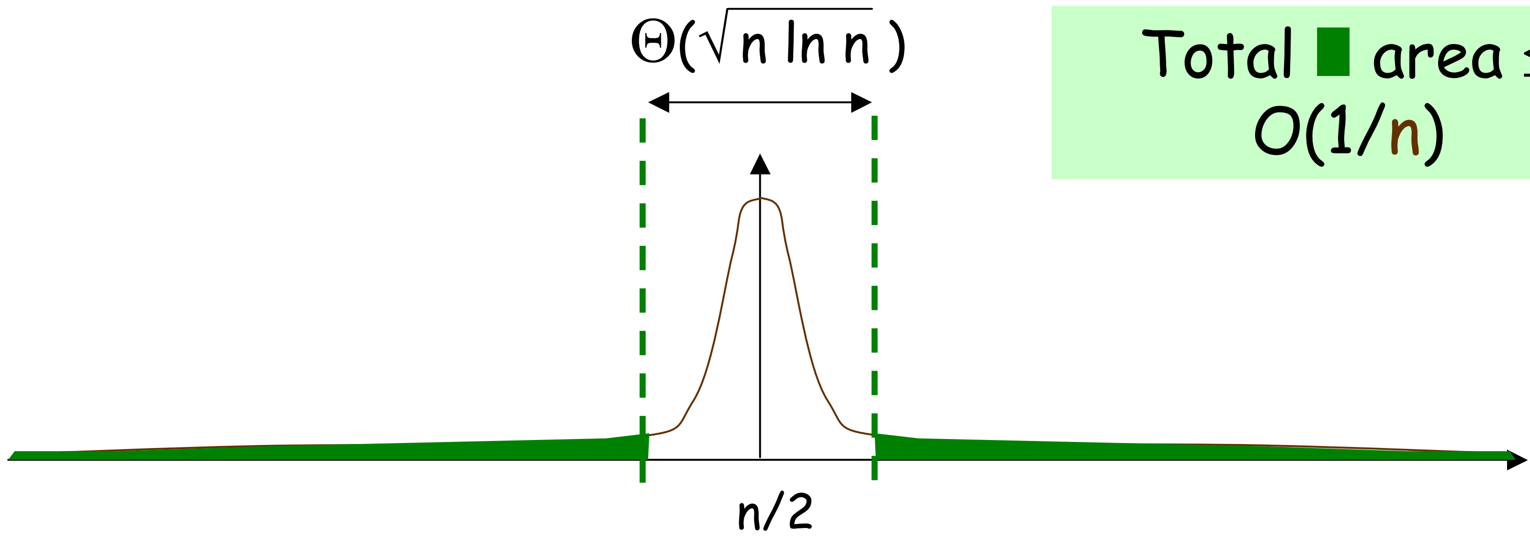
In fact, w.h.p., #heads is around the mean:

$\Pr(|X - n/2| \geq (6n \log_e n)^{0.5}/2)$

$= \Pr(|X - n/2| \geq ((6n \log_e n)^{0.5}/n)(n/2))$

$\leq 2 \exp\{-(1/3)(n/2)((6n \log_e n)/n^2)\}$

$= 2 \exp\{-\log_e n\} = 2/n$



Total ■ area ≤ O(1/n)