CS5314
Randomized Algorithms

Lecture 1: Events and Probability

Objectives

- Unlike other CS courses, this course is a MATH course…
- We will look at a lot of definitions, theorems and proofs
- This lecture will quickly review basic set theory and introduce formal definition of probability

Set

- A set is a group of items
- One way to describe a set: list every item in the group inside { }
  - E.g., { 12, 24, 5 } is a set with three items
- When the items in the set has trend: use …
  - E.g., { 1, 2, 3, 4, … } means the set of natural numbers
- Or, state the rule
  - E.g., $\{ n \mid n = m^2$ for some positive integer $m \}$ means the set { 1, 4, 9, 16, 25, … }

Set

- A set with no items is an empty set denoted by {} or $\emptyset$
- The order of describing a set does not matter
  - { 12, 24, 5 } = { 5, 24, 12 }
- Repetition of items does not matter too
  - { 5, 5, 5, 1 } = { 1, 5 }
- Membership symbol $\in$
  - $5 \in \{ 12, 24, 5 \}$ $7 \notin \{ 12, 24, 5 \}$

Set (Quick Quiz)

- How many items are in each of the following set?
  - $\{ 3, 4, 5, \ldots, 10 \}$
  - $\{ 2, 3, 3, 4, 4, 2, 1 \}$
  - $\{ 2, \{2\}, \{\{2\}\} \}$
  - $\emptyset$
  - $\{\emptyset\}$

Set

Given two sets A and B

- we say $A \subseteq B$ (read as A is a subset of B) if every item in A also appears in B
  - E.g., A = the set of primes, B = the set of integers
- we say $A \subsetneq B$ (read as A is a proper subset of B) if $A \subseteq B$ but $A \neq B$

Warning: Don't be confused with $\in$ and $\subseteq$

- Let $A = \{ 1, 2, 3 \}$. Is $\emptyset \in A$? Is $\emptyset \subseteq A$?

Union and Intersection

Given two sets A and B

- $A \cup B$ (read as the **union** of A and B) is the set obtained by combining all elements of A and B in a single set
  - E.g., $A = \{ 1, 2, 4 \}$ $B = \{ 2, 5 \}$
    $A \cup B = \{ 1, 2, 4, 5 \}$
- $A \cap B$ (read as the **intersection** of A and B) is the set of common items of A and B
  - In the above example, $A \cap B = \{ 2 \}$
- If $A \cap B = \{ \}$, then we say A and B are **disjoint**

Set

- The power set of A is the set of all subsets of A, denoted by 2^A
  - E.g., $A = \{ 0, 1 \}$
    $2^A = \{ \{\}, \{0\}, \{1\}, \{0,1\} \}$
  - How many items in the above power set of A?
- If A has n items, how many items does its power set contain? Why?

Experiment and Sample Space

Experiment : a process producing an outcome

Random experiment : an experiment whose outcome is not known until it is observed

Sample space of a random experiment : the set of all possible outcomes

Event : A subset of the sample space
(called a simple event if there is only 1 element)

Experiment and Sample Space (2)

Example 1:

Experiment:

Throw a die once and observe result

Sample Space: { 1, 2, 3, 4, 5, 6 }

Example 2:

Experiment:

Throw a coin repeatedly till Head is up

Sample Space: ??

Definition of Probability

Given a random experiment, we can talk about its probability space, which consists of three things:

(1) The sample space Ω

(2) The allowable events (any subset of Ω)

(3) The probability function, Pr, which maps any event to a real number such that …

Definition of Probability (2)

... it satisfies the following conditions:

(i) For any event E, $0 \leq Pr(E) \leq 1$

(ii) $Pr(\Omega) = 1$

(iii) For any finite or countably infinite sequence of events $E_1, E_2, \ldots$, if they are pairwise mutually disjoint, then

$$Pr(E_1 \cup E_2 \cup E_3 \ldots)$$

$$= Pr(E_1) + Pr(E_2) + Pr(E_3) + \ldots$$

Example

Experiment:

Throw a die once and observe result

Sample Space: { 1, 2, 3, 4, 5, 6 }

The event {1} corresponds to the case where the observed value is 1.

What is meant by the event {1,2} ?

Example (cont)

Questions:

1. Suppose the die is a fair die,

   so that Pr(1)= Pr(2) = … = Pr(6).

   What is Pr(1)? Why?

2. If Pr(1) = 0.2, Pr(2) = 0.3, Pr(3) = 0.4,

   Pr(4) = 0.1, Pr(5) = Pr(6) = 0.

   Can we obtain Pr({1,2,4})?

A simple lemma

Lemma: For any two events E_1 and E_2

$$Pr(E_1 \cup E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cap E_2)$$

Proof: Let $A=E_1-(E_1 \cap E_2)$ and $B= E_2-(E_1 \cap E_2)$.

Then,

$$Pr(E_1 \cup E_2) = Pr(A) + Pr(B) + Pr(E_1 \cap E_2),$$

$$Pr(E_1) = Pr(A) + Pr(E_1 \cap E_2), \text{ and}$$

$$Pr(E_2) = Pr(B) + Pr(E_1 \cap E_2),$$

so the lemma follows.

Union Bound

Lemma: For any finite and countably infinite sequence of events $E_1, E_2, E_3, \ldots$

$$Pr(E_1 \cup E_2 \cup E_3 \ldots) \leq Pr(E_1)+Pr(E_2)+Pr(E_3)+\ldots$$

How to prove??