CS5314 Randomized Algorithms

Homework 3

Due: 2:10 pm, December 1, 2011 (before class)

1. Let Z be a Poisson random variable with mean μ, where $\mu \geq 1$ is an integer.
   (a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.
   (b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

2. In Page 15 of Lecture Notes 14, we showed that for any nonnegatvie function f,
$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} = m).$$
   (a) Now, suppose we further know that $E[f(X_1^{(m)}, \cdots, X_n^{(m)})]$ is monotonically increasing in m. Show that
$$E[f(Y_1^{(m)}, \cdots, Y_n^{(m)})] \geq E[f(X_1^{(m)}, \cdots, X_n^{(m)})]\Pr(\sum Y_i^{(m)} \geq m).$$
   (b) Combining part (a) with the results in Question 1, prove the monotonically increasing case of the theorem in Page 20 of Lecture Notes 14.

3. Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = E[X]$. During the class, we have learnt that for any $\delta > 0$,
$$\Pr(X \geq (1+\delta)\mu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\mu.$$
   In fact, the above inequality holds for the *weighted* sum of Poisson trials. Precisely, let $a_1, \ldots, a_n$ be real numbers in $[0, 1]$. Let $W = \sum_{i=1}^n a_i X_i$ and $\nu = E[W]$. Then, for any $\delta > 0$,
$$\Pr(W \geq (1+\delta)\nu) < \left(\frac{e^\delta}{(1+\delta)^{(1+\delta)}}\right)^\nu.$$
   (a) Show that the above bound is correct.
   (b) Prove a similar bound for the probability $\Pr(W \leq (1-\delta)\nu)$ for any $0 < \delta < 1$.

4. Let X be a Poisson random variable with mean μ, representing the number of criminals in a city. There are two types of criminals: For the first type, they are not too bad and are reformable. For the second type, they are flagrant.

   Suppose each criminal is independently reformable with probability p (so that flagrant with probability $1 - p$). Let Y and Z be random variables denoting the number of reformable criminals and flagrant criminals (respectively) in the city.

   Show that Y and Z are independent Poisson random variables.

5. Consider assigning some balls to n bins as follows: In the first round, each ball chooses a bin independently and uniformly at random. After that, if a ball lands at a bin by itself, the ball is *served* immediately, and will be removed from consideration. For the number of bins, it remains unchanged.

   In the subsequent rounds, we repeat the process to assign the remaining balls to the bins. We finish when every ball is served.

(a) Suppose at the start of some round b balls are still remaining. Let $f(b)$ denote the expected number of balls that will remain after this round. Given an explicit formula for $f(b)$.

(b) Show that $f(b) \leq b^2/n$.

Hint: You may use *Bernoulli's inequality*:

$$\forall r \in \mathbb{N} \text{ and } x \geq -1, \qquad (1+x)^r \geq 1 + rx.$$

(c) Suppose that every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\log \log n)$ rounds.

6. (Challenging: No marks) Let Z be a Poisson random variable with mean μ, where $\mu \geq 1$ is an integer. This question is to show that $\Pr(Z \leq \mu) \geq 1/2$.

   (a) Show that $\Pr(Z = \mu - h) \geq \Pr(Z = \mu + h + 1)$ for $0 \leq h \leq \mu$.

   (b) Determine a lower bound on $\Pr(Z = \mu - h) - \Pr(Z = \mu + h + 1)$.

   (c) Determine an upper bound on $\Pr(Z \geq 2\mu + 2)$.

   (d) Using parts (a) to (c), show that $\Pr(Z \leq \mu) \geq 1/2$.

7. (No marks) Prove the monotonically decreasing case of the theorem in Page 20 of Lecture Notes 14.