CS5314 Randomized Algorithms

Assignment 1
Due: 2:10 pm, Oct 20, 2011 (before class)

1 Questions

1. A box contains some number of white and black balls. When two balls are drawn without replacement, the probability that both are white is $\frac{1}{3}$.

   (a) Show that
   $$\left(\frac{a-1}{a+b-1}\right)^2 < \frac{1}{3} < \left(\frac{a}{a+b}\right)^2.$$

   (b) Show that
   $$\frac{(\sqrt{3}+1)b}{2} < a < 1 + \frac{(\sqrt{3}+1)b}{2}.$$

   (c) Find the smallest number of balls in the box.

   (d) How small can the total number of balls be if black balls are even in number?

2. Consider the following balls-and-bin game. We start with one black ball and one white ball in a bin. We repeatedly do the following: Choose one ball from the bin uniformly at random, and then put the ball back in the bin with another ball of the same color. (That is, after each step, we have one more ball in the bin.) We repeat until there are n balls in the bin. Show that the number of white balls is equally likely to be any number between 1 and $n-1$.

3. Suppose that a fair coin is flipped n times. For $k > 0$, show that the probability of obtaining a sequence of $\log_2 n + k$ consecutive heads is at most $1/2^k$.

4. Let n be the number of vertices of a graph. Show that the number of distinct min-cut sets in the graph is at most $n(n-1)/2$.

5. The following problem is known as the Monty Hall problem, after the host of the game show "Let's Make a Deal". There are three curtains. Behind one curtain is a new car, and behind the other two are goats. The game is played as follows. The contestant chooses the curtain that she thinks the car is behind. Monty then opens one of the other curtains to show a goat. (Monty may have more than one goat to choose from; in this case, assume he chooses which goat to show uniformly at random.) The contestant can then stay with the curtain she originally chose, or switch to the other unopened curtain. After that, the location of the car is revealed, and the contestant wins the car or the remaining goat. Should the contestant switch curtains or not, or does it make no difference?

6. (a) Show that if E_1 and E_2 are mutually independent, then so are $\overline{E_1}$ and $\overline{E_2}$.

   (b) Show that if $E_1, E_2, \ldots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$.

7. Let X and Y be independent geometric random variables, where X has parameter p and Y has parameter q.

(a) What is the probability that $X = Y$?

(b) What is $\mathrm{E}[\max(X, Y)]$?

(c) What is $\Pr(\min(X, Y) = k)$?

(d) What is $\mathrm{E}[X \mid X \leq Y]$?

2 Further Studies (No marks)

1. During the class, we have studied a simple randomized algorithm so that given any graph, we can find its min-cut with probability at least $2/(n(n-1))$. Now, we define an r-cut of a graph G to be a set of edges in G whose removal will break G into r or more connected components. (That is, the normal definition of a cut is equivalent to a 2-cut here.) Describe a randomized algorithm for finding an r-cut with minimum number of edges. Also, analyze the probability that the algorithm succeeds in one iteration.