1. Let $G$ be a random graph drawn from the $G_{n,p}$ model.
   (a) (10%) What is the expected number of 5-clique in $G$?
   (b) (10%) What is the expected number of 5-cycle in $G$?

2. Suppose we have a set of $n$ vectors, $v_1, v_2, \ldots, v_n$, in $\mathbb{R}^m$. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all $i$. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that
   \[
   \|\rho_1v_1 + \rho_2v_2 + \cdots + \rho_nv_n\| \leq \sqrt{n}.
   \]
   This result shows that if we are allowed to “reflect” each $v_i$ as we wish (i.e., by replacing $v_i$ by $-v_i$), then it is possible that the resultant vector formed by the sum of the $n$ vectors has length at most $\sqrt{n}$.
   (a) (10%) Let $V = \rho_1v_1 + \rho_2v_2 + \cdots + \rho_nv_n$, and recall that
   \[
   \|V\|^2 = V \cdot V = \sum_{i,j} \rho_i\rho_j v_i \cdot v_j.
   \]
   Suppose that each $\rho_i$ is chosen uniformly at random to be -1 or +1. Show that
   \[
   E[\|V\|^2] = n.
   \]
   Hint:
   • What is the value of $E[\rho_i\rho_j]$ when $i \neq j$?
   • What is the value of $E[\rho_i\rho_i]$?
   • What is the value of $v_i \cdot v_i$?
   (b) (5%) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.
   (c) (5%) Your friend, Peter, is more ambitious, and asks if it is possible to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| < \sqrt{n}$ instead of $\|V\| \leq \sqrt{n}$ we have just shown. Give a counter-example why this may not be possible.

3. (40%) Consider a graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that
   \[
   \Pr(X \geq 1) \leq 1/6
   \]
   and that
   \[
   \lim_{n \to \infty} \Pr(X \geq 1) \geq 1/7
   \]
   Hint: Use the conditional expectation inequality.

4. (20%) Use the general form of the Lovasz local lemma to prove that the symmetric version can be improved by replacing the condition $4dp \leq 1$ by the weaker condition $ep(d+1) \leq 1$. 