1. (25%) Let $X$ be a Poisson random variable with mean $\mu$, representing the number of errors on a page of this book. Each error is independently a grammatical error with probability $p$ and a spelling error with probability $1-p$. If $Y$ and $Z$ be random variables representing the number of grammatical and spelling errors (respectively) on a page of this book, prove that $Y$ and $Z$ are Poisson random variables with means $\mu p$ and $\mu (1-p)$, respectively. Also prove that $Y$ and $Z$ are independent.

2. (25%) Let $Z$ be a Poisson random variable of mean $\mu$, where $\mu \geq 1$ is an integer.

(a) Show that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ for $0 \leq h \leq \mu - 1$.

(b) Using part (a), argue that $\Pr(Z \geq \mu) \geq 1/2$.

Hint: Use the definition of Poisson distribution $\Pr(Z = j) = e^{-\mu} \mu^j / j!$.

3. (25%) In Page 15 of Lecture Notes 14 we showed that, for any nonnegative functions $f$,

$$
\mathbb{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})] \geq \mathbb{E}[f(X_1^{(m)}, ..., X_n^{(m)})] \Pr(\sum Y_i^{(m)} = m)
$$

(a) Now suppose we further know that $\mathbb{E}[f(X_1^{(m)}, ..., X_n^{(m)})]$ is monotonically increasing in $m$. Show that

$$
\mathbb{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})] \geq \mathbb{E}[f(X_1^{(m)}, ..., X_n^{(m)})] \Pr(\sum Y_i^{(m)} \geq m)
$$

(b) Combining part (a) with the result in Question 2, prove the following theorem.

**Theorem:** Let $f(x_1, ..., x_n)$ be a nonnegative function such that $\mathbb{E}[f(X_1^{(m)}, ..., X_n^{(m)})]$ is monotonically increasing in $m$. Then

$$
\mathbb{E}[f(X_1^{(m)}, ..., X_n^{(m)})] \leq 2\mathbb{E}[f(Y_1^{(m)}, ..., Y_n^{(m)})]
$$

4. (25%) We consider another way to obtain Chernoff-like bound in the balls-and-bins setting. Consider $n$ balls thrown randomly into $n$ bins. Let $X_i = 1$ if the $i$-th bin is empty and 0 otherwise. Let $X = \sum_{i=1}^{n} X_i$.

Let $Y_i$ be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$. Let $Y = \sum_{i=1}^{n} Y_i$.

(a) Show that $\mathbb{E}[X_1 X_2 \cdots X_k] \leq \mathbb{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \geq 1$.

(b) Show that $X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} = X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k}$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.

(c) Show that $\mathbb{E}[e^{tX}] \leq \mathbb{E}[e^{tY}]$ for all $t \geq 0$.

*Hint:* Use the expansion for $e^x$ and compare $\mathbb{E}[e^{tX}]$ to $\mathbb{E}[e^{tY}]$.

(d) Derive a Chernoff bound for $\Pr(X \geq (1 + \delta)\mathbb{E}[X])$. 
