# Randomized algorithm

## Tutorial 6

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# Solution to assignment 4

[Question 1]: Let $G$ be a random graph drawn from the $G_{n,p}$ model.

1. (10%) What is the expected number of 5-clique in $G$?
2. (10%) What is the expected number of 5-cycle in $G$?

[Solution]:

1. Number of 5-vertex set: $\binom{n}{5}$
   Probability of a 5-clique : $p^{10}$
   $\mathrm{E}[\text{number of 5-clique in } G] = \binom{n}{5} \cdot p^{10}$.
2. Each 5-vertex set can compose 5! cycles but each cycle can be represented in 10 different ways.
   Ex: $\{1,2,3,4,5\}$ $\{2,3,4,5,1\}$ $\{3,4,5,1,2\}$ $\{4,5,1,2,3\}$ $\{5,1,2,3,4\}$
   The total number of 5-cycle is $\binom{n}{5} \cdot \frac{5!}{10} = 12\binom{n}{5}$

$$\mathrm{E}[\text{number of 5-cycle in } G] = 12\binom{n}{5}p^5$$

[Question 2]: Suppose we have a set of $n$ vectors, $v_1, v_2, \ldots, v_n$, in $R^m$. Each vector is of unit-length, i.e., $\|v_i\| = 1$ for all $i$. In this question, we want to show that, there exists a set of values, $\rho_1, \rho_2, \ldots, \rho_n$, each $\rho_i \in \{-1, +1\}$, such that

$$\|\rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n\| \leq \sqrt{n}.$$

Intuitively, if we are allowed to "reflect" each $v_i$ as we wish (i.e., by replacing $v_i$ by $-v_i$), then it is possible that the vector formed by the sum of the $n$ vectors is at most $\sqrt{n}$ long.

1. Let $V = \rho_1 v_1 + \rho_2 v_2 + \cdots + \rho_n v_n$, and recall that

$$\|V\|^2 = V \cdot V = \sum_{i,j} \rho_i \rho_j v_i \cdot v_j.$$

   Suppose that each $\rho_i$ is chosen uniformly at random to be -1 or $+1$. Show that

$$\mathrm{E}[\|V\|^2] = n.$$

2. (5%) Argue that there exists a choice of $\rho_1, \rho_2, \ldots, \rho_n$ such that $\|V\| \leq \sqrt{n}$.
3. (5%) Your friend, Peter, is more ambitious, and asks if it is possible to to choose $\rho_1, \rho_2, \ldots, \rho_n$ such that

$$\|V\| < \sqrt{n}$$

   instead of $\|V\| \leq \sqrt{n}$ we have just shown. Give a counter-example why this may not be possible.

[Solution 1]:
Given that

$$\|V\|^2 = V \cdot V,$$

we have

$$\mathrm{E}[\|V\|^2] = \mathrm{E}[V \cdot V] = \mathrm{E}\left[\sum_{i,j} \rho_i \rho_j v_i \cdot v_j\right].$$

When $i \neq j$, $\rho_i$ and $\rho_j$ are independent random variables, so that

$$\mathrm{E}[\rho_i \rho_j] = \mathrm{E}[\rho_i]\mathrm{E}[\rho_j] = 0.$$

On the other hand, when $i = j$, we have

$$\mathrm{E}[\rho_i \rho_i] = 0.5(1)^2 + 0.5(-1)^2 = 1.$$

Also, $v_i \cdot v_i = v_i^2 = 1$ since $v_i$ is a unit vector. Combining the above, we obtain

$$\begin{aligned} \mathrm{E}[\|V\|^2] &= \mathrm{E}\left[\sum_{i,j} \rho_i \rho_j v_i \cdot v_j\right] \\ &= \mathrm{E}\left[\sum_{i} \rho_i \rho_i v_i \cdot v_i\right] + \mathrm{E}\left[\sum_{i \neq j} \rho_i \rho_j v_i \cdot v_j\right] \\ &= \sum_{i} \mathrm{E}\left[\rho_i \rho_i\right] + \sum_{i \neq j} \mathrm{E}\left[\rho_i \rho_j v_i \cdot v_j\right] \\ &= n + 0 = n. \end{aligned}$$

[Solution 2]:

Since $\mathrm{E}[\|V\|^2] = n$, there must be a choice of reflection $\rho$'s such that $\|V\|^2$ is at most $n$. With that particular choice, $\|V\| \leq \sqrt{n}$.

[Solution 3]:
Set $v_1 = (1,0)$ and $v_2 = (0,1)$. No matter what $\rho_1$ and $\rho_2$ is, $\rho_1\rho_2 v_1 \cdot v_2 = \sqrt{2}$.
In general in $\mathbb{R}^m$, we can set

$$v_k = (\underbrace{0,\dots,0}_{k-1 \text{ zeroes}}, 1, 0, \dots, 0)$$

for $k = 1$ to $m$. Then under any choice of the reflections, the resulting vector $V$ will have length exactly $\sqrt{m}$.

[Question 3]: Let $F$ be a family of subsets of $N = \{1, 2, ..., n\}$. $F$ is called an antichain if there are no $A, B \in F$ satisfying $A \subset B$.
Ex:
$N = \{1, 2, 3\}$
All subsets of $N$ :$\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
$\{\{1, 3\}, \{1, 2\}, \{2, 3\}\}$ is an antichain.
$\{\{1, 2, 3\}, \{1, 2\}\}$ is not an antichain. ($\{1, 2\} \subset \{1, 2, 3\}$)

We want to prove that $|F| \leq \binom{n}{\lfloor n/2 \rfloor}$. How?
Let $\sigma \in S_n$ be a random permutation of the elements of $N$ and consider the random variable

$$X = |\{i : \{\sigma(1), \sigma(2), ...\sigma(i)\} \in F\}|$$

What is $X$?

Let us set $F = \{\{2\}, \{1, 3\}\}$.
For $\sigma = \{2, 3, 1\}$, $X = 1$. $\because$ $\{2\} \in F$.
For $\sigma = \{3, 2, 1\}$, $X = 0$. $\because$ None of $\{3\}$, $\{3, 2\}$ ,$\{3, 2, 1\}$ are in $F$.
For $\sigma = \{3, 1, 2\}$, $X = 1$. $\because$ $\{1, 3\} \in F$.
Try to find the relation between $X$ and $F$.

[Solution]:
Partition $F$ into $n$ sets, $k_1, k_2, \cdots k_n$ where $k_i$ is the number of those sets whose size is $i$. Therefore, $|F| = k_1 + k_2 + \cdots + k_n$. The expected number of $X$ is the can be considered as all the possible case in all permutations.

$$\begin{aligned}
\mathrm{E}[X] &= (k_1 \cdot (n-1)! \cdot 1! + k_2 \cdot (n-2)! \cdot 2! + \cdots + k_n \cdot n!)/n! \\
&= \frac{k_1}{\binom{n}{1}} + \frac{k_2}{\binom{n}{2}} + \cdots + \frac{k_n}{\binom{n}{n}} \\
&\geq \frac{(k_1 + k_2 + \cdots + k_n)}{\binom{n}{n/2}} \\
&= \frac{|F|}{\binom{n}{n/2}}
\end{aligned}$$

As we have already known that $X$ is an indicator random variable, $\mathrm{E}[X] \leq 1$. Thus, $\frac{|F|}{\binom{n}{n/2}} \leq 1$.

[Question 4]: Consider a graph in $G_{n,p}$, with $p = 1/n$. Let $X$ be the number of triangles in the graph, where a triangle is a clique with three edges. Show that

$$Pr(X \geq 1) \leq 1/6$$

and that

$$\lim_{n \to \infty} Pr(X \geq 1) \geq 1/7$$

[Solution 1]:

$$\begin{aligned}
\Pr(X \geq 1) &= \binom{n}{3} p^3 \\
&= \frac{n(n-1)(n-2)}{6n^3} \\
&\leq \frac{1}{6}
\end{aligned}$$

[Solution 2]: Consider all possible size-3 subsets of the vertices of the random graph generated by $G_{n,p}$. Let $X_i$ be an indicator that denote $i$th size-3 subset forms a triangle in $G$.
Now we consider $E[X|X_i = 1]$.

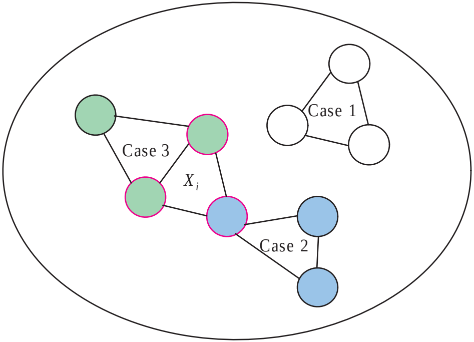



Figure: 4 cases. Case 1:$X_i$; Case 2: $p^3$; Case 3: $p^3$; Case 4: $p^2$.

By conditional expectation inequality, $\Pr(X) = \Sigma_{i=1}^{\binom{n}{3}} \frac{Pr(X_i=1)}{\mathrm{E}[X|X_i=1]}$

$$\begin{aligned}
\mathrm{E}[X|X_i = 1] &= 1 + \binom{n-3}{3}p^3 + 3\binom{n-3}{2}p^3 + 3\binom{n-3}{1}p^2 \\
\Pr(X \geq 1) &= \frac{\binom{n}{3}p^3}{1 + \binom{n-3}{3}p^3 + 3\binom{n-3}{2}p^3 + 3\binom{n-3}{1}p^2} \\
&= \frac{1/6}{1 + 1/6 + 0 + 0} = \frac{1}{7}
\end{aligned}$$

$$\lim_{n \to \infty} Pr(X \geq 1) \geq 1/7$$

# Hint for assignment 5

[Question 1]: Consider the Markov chain in the following figure. Suppose that $n \geq 4$. Find the expected number of moves to reach $n$ starting from position $i$, when (a) $i = n - 1$, (b) $i = n - 2$, (c) $i = n - 3$, and (d) $i = n - 4$. (Express the answers exactly in terms of $n$.)

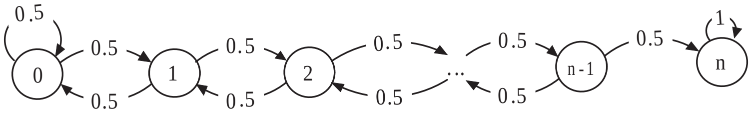



Figure: The modeling Markov chain for 2-SAT problem.

[Question 2]: Consider the Markov chain in the following figure.

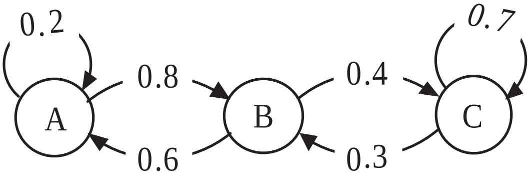
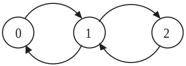



Figure: The modeling Markov chain of this question.

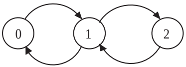
1. (20%) Argue that the Markov chain is aperiodic and irreducible.
2. (20%) Find the stationary probability.
3. (20%) Find the probability of being in state 0 after 16 steps if the chain begins at state chosen uniformly at random from the 3 states.

[Hint]:

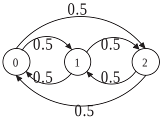
- Irreducible: Every state $j$ can reach every state $k$




- Periodic: Once we start at state $j$, we can only return to $j$ after a multiple of $d$ steps.




- Stationary distribution: A probability distribution that $p(n)$ remains a certain distribution when $n > t$.




$p(t)=\{1/3, 1/3, 1/3\}$

Thank you