Randomized algorithm

Tutorial 4

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Solution to Midterm

Question 1

Question 2

Hint for assignment 3

Question 1

Question 2

Question 3

Question 4

Question 5

Solution to Midterm

[Question 1]: Alice and Bob decide to have children until either they have their first girl or they have $k \geq 1$ children. Assume that each child is a boy with probability $1/3$.

1. What is the expected number of girls that they have?
2. What is the expected number of boys that they have?
3. Suppose Alice and Bob simply decide to keep having children until they have their first girl. What is the expected number of boys that they have?

$\mathrm{E}[X]$: the expected number of girls.
$\mathrm{E}[Y]$: the expected number of boys.
$\mathrm{E}[Z]$: the expected number of all children.

$$\sum_{k=1}^{n} a_i * r^k = \frac{a_i(1 - r^{n+1})}{1 - r}$$

[Solution]:

$$\mathrm{E}[X] = \frac{1}{3^k} * 0 + (1 - \frac{1}{3^k}) * 1 = 1 - \frac{1}{3^k}$$

$$\begin{aligned}
\mathrm{E}[Z] &= 1 * \frac{2}{3} + 2 * \frac{1}{3} * \frac{2}{3} + \cdots + k * \frac{1}{3^{k-1}} * \frac{2}{3} + k * \frac{1}{3^k} \\
&= \left(\frac{2}{3} + \frac{2}{3^2} + \cdots + \frac{2}{3^k}\right) + \left(\frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^k}\right) + \\
&\quad \cdots + \left(\frac{2}{3^{k-1}} + \frac{2}{3^k}\right) + \frac{2}{3^k} + k * \frac{1}{3^k}
\end{aligned}$$

$$
\begin{aligned}
\mathrm{E}[Z] &= \left(1-\frac{1}{3^k}\right)+\left(\frac{1}{3^1}-\frac{1}{3^k}\right)+\cdots+\left(\frac{1}{3^{k-1}}-\frac{1}{3^k}\right)+k*\frac{1}{3^k} \\
&= \left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{k-1}}-\frac{k}{3^k}\right)+k*\frac{1}{3^k} \\
&= \left[\frac{3}{2}\left(1-\frac{1}{3^k}\right)-\frac{k}{3^k}\right]+k*\frac{1}{3^k} \\
&= \frac{3}{2}-\frac{1}{2}*\frac{1}{3^{k-1}}
\end{aligned}
$$

$\mathrm{E}[Y]=\mathrm{E}[Z]-\mathrm{E}[X]=\frac{1}{2}-\frac{1}{2}*\frac{1}{3^k}$

$\lim_{k\to\infty}\frac{1}{2}-\frac{1}{2}*\frac{1}{3^k}=\frac{1}{2}$

[Question 2]: A permutation $\pi : [1, n] \rightarrow [1, n]$ can be represented as a set of cycles as follows. Let there be one vertex for each number $i, i = 1, \ldots, n$. If the permutation maps the number i to the number $\pi(i)$, then a directed arc is drawn from vertex i to vertex $\pi(i)$. This leads to a graph that is a set of disjoint cycles. Notice that some of the cycles could be self-loop. What is the expected number of cycles in a random permutation if n numbers?

[Solution]: The total number of permutation that v is in a cycle of exactly length k, is equal to:

$$\binom{n-1}{k-1}(k-1)!(n-k)! = (n-1)!$$

which is independent of k.
Thus, the probability that v is in a cycle of length k is $(n-1)!/n! = 1/n$.
Let Y denote the number of cycles in the graph, and let Y_i be a random variable such that

$$Y_i = \frac{1}{\text{length of cycle where vertex } i \text{ belongs to}}$$

Since $Y = \sum_{i=1}^{n} Y_i$, we have

$$
\begin{aligned}
\mathrm{E}[Y] &= \sum_{i=1}^{n} \mathrm{E}[Y_i] \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{1}{j} \Pr\left(Y_i = \frac{1}{j} \right) \right) \\
&= \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} \frac{1}{j} \right) \\
&\approx \sum_{i=1}^{n} \frac{1}{n} \times \ln n = \ln n
\end{aligned}
$$

Hint for assignment 3

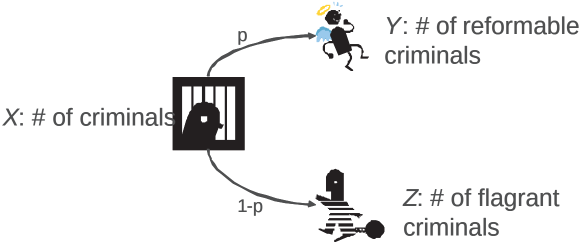
[Question 1]: Let X be a Poisson random variable with mean λ.

1. What is the most likely value of X
   1.1 when λ is an integer?
   1.2 when λ is not an integer?

   Hint: Compare $\Pr(X = k + 1)$ with $\Pr(X = k)$.
2. We define the median of X to be the least number m such that $\Pr(X \leq m) \geq 1/2$. What is the median of X when $\lambda = 3.9$?

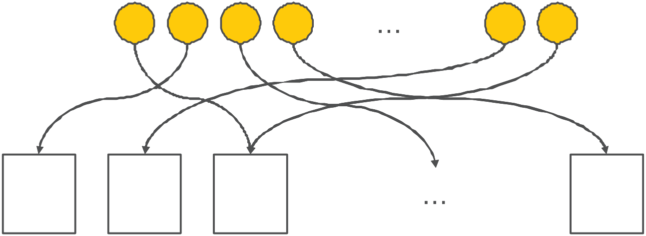
[Question 2]: Let X be a Poisson random variable with mean μ, representing the number of criminals in a city. There are two types of criminals: For the first type, they are not too bad and are reformable. For the second type, they are flagrant. Suppose each criminal is independently reformable with probability p (so that flagrant with probability $1 - p$). Let Y and Z be random variables denoting the number of reformable criminals and flagrant criminals (respectively) in the city. Show that Y and Z are independent Poisson random variables.

[Hint]:

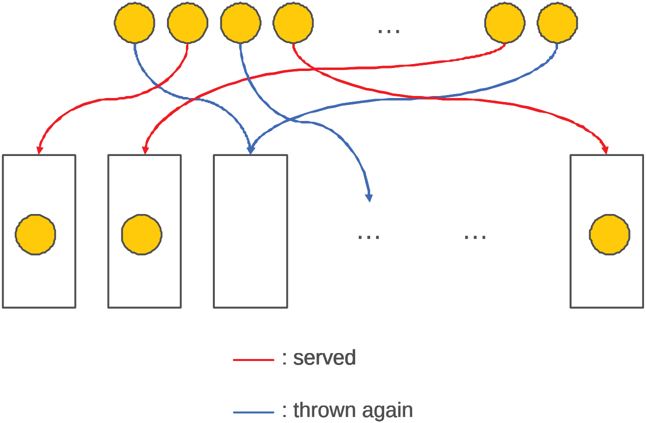


By definition of Poisson random variable with some condition. Try to show $\Pr(Y = k) =$? and $\Pr(Z = k) =$?

[Question 3]: Consider assigning some balls to n bins as follows: In the first round, each ball chooses a bin independently and uniformly at random.



After that, if a ball lands at a bin by itself, the ball is *served* immediately, and will be removed from consideration. For the number of bins, it remains unchanged.



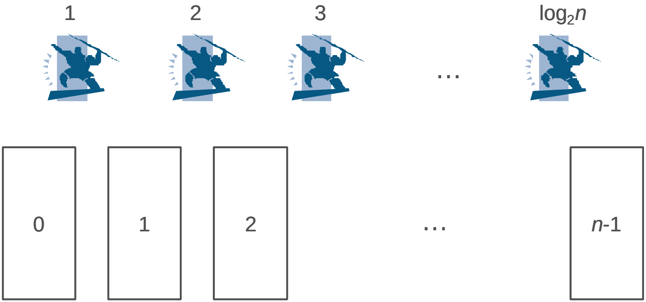
In the subsequent rounds, we repeat the process to assign the remaining balls to the bins. We finish when every ball is served.

1. Suppose at the start of some round b balls are still remaining. Let $f(b)$ denote the expected number of balls that will remain after this round. Given an explicit formula for $f(b)$.
2. Show that $f(b) \leq b^2/n$.
   Hint: You may use *Bernoulli's inequality*:
   $$\forall r \in \mathbb{N} \text{ and } x \geq -1, \qquad (1+x)^r \geq 1 + rx.$$
3. Suppose we have $\frac{n}{k}$ balls initially, for some fixed constant $k > 1$. Every round the number of balls served was exactly the expected number of balls to be served. Show that all the balls would be served in $O(\log \log n)$ rounds.

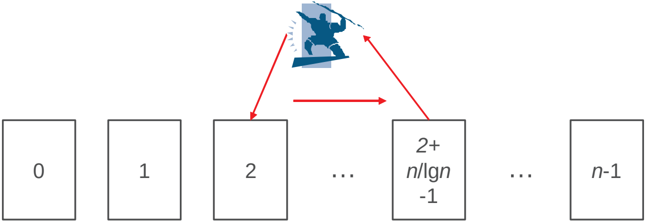
[Hint]:

1. E[number of bins with 1 ball] $=$?
2. *Bernoulli's inequality*
3. The number of balls at each round decreased exactly according to expectation: $m, f(m), f(f(m)), f(f(f(m)))$.

[Question 4]: Suppose that we vary the balls-and-bins process as follows. For convenience let the bins be numbered from 0 to $n-1$. There are $\log_2 n$ players.



Each player chooses a starting location ℓ uniformly at random from $[0, n-1]$ and then places one ball in each of the bins numbered $\ell \bmod n, \ell+1 \bmod n, \ldots, \ell + n/\log_2 n - 1 \bmod n$.(Assume that n is a multiple of $\log_2 n$.)



Show that the maximum load in this case is only $O(\log\log n/\log\log\log n)$ with probability that approaches 1 as $n \longrightarrow \infty$.

[Hint]: Total number of balls are n. How would the probability for bin 1 to receive at least M balls?

[Question 5]: We consider another way to obtain Chernoff-like bound in the balls-and-bins setting without using the theorem in Page 13 of Lecture 14.
Consider n balls thrown randomly into n bins. Let $X_i = 1$ if the i-th bin is empty and 0 otherwise. Let $X = \Sigma_{i=1}^{n} X_i$.
Let Y_i be independent Bernoulli random variable such that $Y_i = 1$ with probability $p = (1 - 1/n)^n$. Let $Y = \Sigma_{i=1}^{n} Y_i$.

1. Show that $\mathrm{E}[X_1 X_2 \cdots X_k] \le \mathrm{E}[Y_1 Y_2 \cdots Y_k]$ for any $k \ge 1$.
2. Show that ${X_1}^{j_1} {X_2}^{j_2} \cdots {X_k}^{j_k} = X_1 X_2 \cdots X_k$ for any $j_1, j_2, \ldots, j_k \in \mathbb{N}$.
3. Show that $\mathrm{E}[e^{tX}] \le \mathrm{E}[e^{tY}]$ for all $t \ge 0$.
   Hint: Use the expansion for e^x and compare $\mathrm{E}[e^{tX}]$ to $\mathrm{E}[e^{tY}]$.
4. Derive a Chernoff bound for $Pr(X \ge (1+\delta)\mathrm{E}[X])$.

[Hint]:
Add oil.

Thank you