Randomized algorithm

Tutorial 3

Joyce

2009-11-10
Solution to Assignment 2

Question 1
Question 2
Question 3
Question 4
Question 5
Question 6
Solution to Assignment 2
A fixed point of a permutation \( \pi : [1, n] \rightarrow [1, n] \) is a value for which \( \pi(x) = x \). Find the variance in the number of fixed points of a permutation chosen uniformly at random from all permutations.
[Solution]: Let $X_i$ be an indicator such that $X_i = 1$ if $\pi(i) = i$. Then, $\sum_{i=1}^{n} X_i$ is the number of fixed points. 

$$\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_iX_j] - (\mathbb{E}[X])^2$$

Since $\mathbb{E}[X_i^2] = \Pr(X_i^2 = 1) = \frac{1}{n}$ and

$$\mathbb{E}[X_iX_j] = \Pr(X_iX_j = 1)$$

$$= \Pr(X_i = 1)\Pr(X_j = 1 \mid X_i = 1)$$

$$= \frac{1}{n(n-1)},$$

we get

$$\text{Var}[X] = n \times \frac{1}{n} + n(n-1) \times \frac{1}{n(n-1)} - 1 = 1.$$
[Question 2]: Recall that the covariance of random variables $X$ and $Y$ is:

$$\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])].$$

We have seen that if $X$ and $Y$ are independent, then the covariance is 0. Interestingly, if $X$ and $Y$ are not independent, the covariance may still be 0.

(15%) Construct an example where $X$ and $Y$ are not independent, yet $\text{Cov}[X, Y] = 0.$
[Solution]: Let the sample space be \{-1, 0, 1\}, with each outcome having the same probability to occur. Let \( X \) denote the outcome and let \( Y = X^2 \). We know that \( \Pr(X = 0 \cap Y = 0) = 1/3 \) and \( \Pr(X = 0)\Pr(Y = 0) = 1/9 \) so that \( X \) and \( Y \) are not independent. However,

\[
\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]
\]
\[
= \frac{1}{3}((-1)(1 - \mathbb{E}[Y]) + (0)(0 - \mathbb{E}[Y]) + (1)(1 - \mathbb{E}[Y]))
\]
\[
= 0.
\]
[Question 3]:
The weak law of large numbers state that, if $X_1, X_2, X_3, \ldots$ are independent and identically distributed random variables with finite mean $\mu$ and finite standard deviation $\sigma$, then for any constant $\epsilon > 0$ we have

$$\lim_{n \to \infty} \Pr \left( \left| \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n} - \mu \right| > \epsilon \right) = 0$$

(15%) Use Chebyshev’s inequality to prove the weak law of large numbers.
[Solution]:

\[
\Pr \left( \left| \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n} - \mu \right| > \epsilon \right)
\]

\[
= \Pr \left( |X_1 + X_2 + X_3 + \ldots + X_n - n\mu| > n\epsilon \right)
\]

By Chebyshev's inequality:

\[
\Pr \left( |X_1 + X_2 + X_3 + \ldots + X_n - n\mu| > n\epsilon \right) \leq \frac{\text{Var}[X_1 + X_2 + X_3 + \ldots + X_n]}{(n\epsilon)^2}
\]

\[
= \frac{\Sigma \text{Var}[X_i]}{n^2\epsilon^2} = \frac{n\sigma^2}{n^2\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}
\]

\[
\therefore \lim_{n \to \infty} \Pr \left( \left| \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n} - \mu \right| > \epsilon \right) = 0
\]
[Question 4]: Suppose you are given a biased coin that has \( \Pr(\text{head}) = p \). Also, suppose that we know \( p \geq a \), for some fixed \( a \). Now, consider flipping the coin \( n \) times and let \( n_H \) be the number of times a head comes up. Naturally, we would estimate \( p \) by the value \( \tilde{p} = n_H / n \).

1. Show that for any \( \epsilon \in (0, 1) \),

\[
\Pr\left( |p - \tilde{p}| > \epsilon p \right) < \exp\left( -\frac{na\epsilon^2}{2} \right) + \exp\left( -\frac{na\epsilon^2}{3} \right)
\]

2. Show that for any \( \epsilon \in (0, 1) \), if

\[
n > \frac{3 \ln(2/\delta)}{a\epsilon^2},
\]

then

\[
\Pr\left( |p - \tilde{p}| > \epsilon p \right) < \delta.
\]
[Solution]: (1)

\[ \Pr(|p - \tilde{p}| > \epsilon p) \]
\[ = \Pr(n\tilde{p} < np - \epsilon pn) + \Pr(n\tilde{p} > np + \epsilon pn) \]
\[ = \Pr(X < E[X](1 - \epsilon)) + \Pr(X > E[X](1 + \epsilon)) \]
\[ < e^{-np\epsilon^2/2} + e^{-np\epsilon^2/3} \]
\[ < e^{-n\epsilon^2/2} + e^{-n\epsilon^2/3} \]
[Solution]: (2)

\[ n > \frac{3 \ln(2/\delta)}{a\epsilon^2} \]

\[ \Rightarrow na\epsilon^2/3 > \ln(2/\delta) \]

\[ \Rightarrow \ln(\delta/2) > -na\epsilon^2/3 \]

\[ \Rightarrow \delta/2 > e^{-na\epsilon^2/3} \]

\[ \Rightarrow \delta > e^{-na\epsilon^2/3} + e^{-na\epsilon^2/2} \]

\[ \Rightarrow \delta > \Pr(|p - \tilde{p}| > \epsilon p) \]
[Question 5]: (20%) Let $X_1, X_2, \ldots, X_n$ be independent Poisson trials such that $\Pr(X_i) = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$. During the class, we have learnt that for any $\delta > 0$,

$$\Pr(X \geq (1 + \delta)\mu) < \left( \frac{e^\delta}{(1 + \delta)(1+\delta)} \right)^\mu$$

In fact, the above inequality holds for the weighted sum of Poisson trials. Precisely, let $a_1, \ldots, a_n$ be real numbers in $[0, 1]$. Let $W = \sum_{i=1}^{n} a_iX_i$ and $\nu = \mathbb{E}[W]$. Then, for any $\delta > 0$,

$$\Pr(W \geq (1 + \delta)\nu) < \left( \frac{e^\delta}{(1 + \delta)(1+\delta)} \right)^\nu$$

1. Show that the above bound is correct.

2. Prove a similar bound for the probability $\Pr(W \leq (1 - \delta)\nu)$ for any $0 < \delta < 1$. 
[Solution]: (1) Since \( W = \sum_{i=1}^{n} a_i X_i \), we have

\[
\nu = \mathbb{E}[W] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i] = \sum_{i=1}^{n} a_i p_i
\]

For any \( i \),

\[
\mathbb{E}[e^{ta_i X_i}] = p_i e^{ta_i} + (1 - p_i) = 1 + p_i(e^{ta_i} - 1) \leq e^{p_i(e^{ta_i} - 1)}
\]

**Claim 1.** For any \( x \in [0, 1] \), \( e^{tx} - 1 \leq x(e^t - 1) \)

\[
f(x) = x(e^t - 1) - e^{tx} + 1
\]

\[
\Rightarrow f'(x) = (e^t - 1) - te^{tx}
\]

\[
\Rightarrow f'(x) = 0 \text{ (when } x = x^* = (\ln(e^t - 1) - \ln t)/t)\]

\[
\Rightarrow f''(x) = -t^2 e^{tx} \leq 0
\]

In other words, for \( x \in [0, 1] \), \( f(x) \) achieves minimum value either at \( f(0) \) or \( f(1) \). So \( f(x) \geq \min\{f(0), f(1)\} = 0 \) for all \( x \in [0, 1] \).
[Solution]: Hence,

\[ E[e^{\eta_i X_i}] \leq e^{p_i(e^{\eta_i} - 1)} \leq e^{a_i p_i(e^t - 1)}, \]

By the independence of \( X_i \)'s and property of MGF,

\[ E[e^{tW}] = \prod_{i=1}^{n} E[e^{\eta_i X_i}] \leq \prod_{i=1}^{n} E[e^{a_i p_i(e^t - 1)}] = e^{\nu(e^t - 1)} \]

For any \( t > 0 \), we have

\[ \Pr(W \geq (1 + \delta)\nu) = \Pr(e^{tW} \geq e^{t(1+\delta)\nu}) = \frac{E[e^{tW}]}{e^{t(1+\delta)\nu}} \leq \frac{e^{\nu(e^t - 1)}}{e^{t(1+\delta)\nu}} \]

Then, for any \( \delta > 0 \), we can set \( t = \ln(1 + \delta) > 0 \) and obtain:

\[ \Pr(W \geq (1 + \delta)\nu) < \left( \frac{e^{\delta}}{(1 + \delta)^{1+\delta}} \right)^\nu \]
[Solution]: (2) For any $t < 0$, we have

$$
\Pr(W \leq (1 - \delta)\nu) = \Pr(e^{tW} \geq e^{t(1-\delta)\nu}) \\
\leq \frac{\mathbb{E}[e^{tW}]}{e^{t(1-\delta)\nu}} \\
\leq \frac{e^{\nu(e^t-1)}}{e^{t(1-\delta)\nu}}
$$

Then, for any $0 < \delta < 1$, we can set $t = \ln(1 - \delta) < 0$ and obtain:

$$
\Pr(W \leq (1 - \delta)\nu) < \left(\frac{e^\delta}{(1 - \delta)^{(1-\delta)}}\right)^\nu
$$
[Question 6]: (30%) Consider a collection $X_1, X_2, \ldots, X_n$ of $n$ independent geometric random variables with parameter $1/2$. Let $X = \sum_{i=1}^{n} X_i$ and $0 < \delta < 1$.

1. By applying Chernoff bound to a sequence of $(1 + \delta)(2n)$ fair coin tosses, show that

$$\Pr(X > (1 + \delta)(2n)) < \exp\left(\frac{-n\delta^2}{2(1 + \delta)}\right).$$
[Solution]: (1)

$X_i$: a sequence of coin flips until the first heads comes.

$\Sigma X_i$: a sequence of coin flips until we see the $n$-th head.

$X > (1 + \delta)2n$: the $n$-th head does not occur among the first $(1 + \delta)2n$ coin flips.

Let $Y$ be the random variable giving the number of heads among the first $(1 + \delta)2n$ coin flips. Then we have

$\Pr(X > (1 + \delta)2n) = \Pr(Y < n)$
Noting that $E[Y] = (1 + \delta)n$, we have

$$\Pr(X > (1 + \delta)2n) = \Pr(Y < n)$$

$$= \Pr(Y < (1 - \frac{\delta}{1+\delta})(1 + \delta)n)$$

$$\leq \exp\left(\frac{- (1 + \delta)n \cdot \frac{\delta^2}{2(1 + \delta)^2}}{2(1 + \delta)} \right)$$

$$= \exp\left(\frac{n\delta^2}{2(1 + \delta)} \right)$$
2. Derive a Chernoff bound on \( \Pr(X > (1 + \delta)(2n)) \) using the moment generating function for geometric random variables as follows:

(i) Show that for \( e^t < 2 \),
\[
E \left[ e^{tX_i} \right] = \frac{e^t}{2 - e^t}.
\]

(ii) Show that for \( t \in (0, \ln 2) \),
\[
\left| \frac{1}{(2 - e^t)e^{t(1+2\delta)}} \right| \text{ is minimized when } t = \ln \left( 1 + \frac{\delta}{1 + \delta} \right).
\]

(iii) Show that
\[
\Pr(X > (1 + \delta)(2n)) < \left( \left( 1 - \frac{\delta}{1 + \delta} \right) \left( 1 + \frac{\delta}{1 + \delta} \right)^{1+2\delta} \right)^{-n}.
\]
[Solution]: (2-i)
Recall that $e^t < 2$. Then we have:

$$E[e^{tX_i}] = \frac{1}{2}e^t + \frac{1}{4}e^{2t} + \frac{1}{8}e^{3t} + ...$$

$$= \sum_{k=1}^{\infty} \left( \frac{e^t}{2} \right)^k$$

$$= \frac{e^t/2}{1 - e^t/2}$$

$$= \frac{e^t}{2 - e^t}$$

(2-ii)
Use differentiation with respect to $t$. 
[Solution](2-iii) Apply Markov inequality,

\[ \Pr(X > (1 + \delta)2n) = \Pr(\exp(tX) > \exp(t(1 + \delta)2n)) \]

\[ \leq \frac{\mathbb{E}[\exp(tX)]}{\exp(t(1 + \delta)2n)} \]

\[ = \frac{\prod \mathbb{E}[\exp(tX_i)]}{\exp(t(1 + \delta)2n)} \]

Combine former results and apply \( t = \ln(1 + \frac{\delta}{1 + \delta}) \),

\[ \Pr(X > (1 + \delta)2n) = \frac{\prod \mathbb{E}[\exp(tX_i)]}{\exp(t(1 + \delta)2n)} \]

\[ \leq \frac{e^{tn}}{(2 - e^t)e^{t(1+\delta)2n}} \]

\[ = \left( \left(1 - \frac{\delta}{1 + \delta}\right) \left(1 + \frac{\delta}{1 + \delta}\right)^{1 + 2\delta}\right)^{-n} \]
3. It is known that when $\delta$ is small, there exists $\epsilon > 0$ such that

$$1 - \frac{\delta}{1 + \delta} > e^{-\epsilon}, \quad \left(1 + \frac{\delta}{1 + \delta}\right)^{(1+\delta)/\delta} > e^{1-\epsilon},$$

and

$$\frac{(1 + 2\delta)\delta}{1 + \delta} > \delta^2.$$

Show that in this case, the bound in 6(b)-(iii) becomes

$$\Pr(X > (1 + \delta)(2n)) < \exp \left( -n(1 - \epsilon)\delta^2 - \epsilon \right).$$

Conclude that when $\delta$ is small enough such that $\epsilon$ is arbitrarily close to 0, the above bound is tighter than the bound obtained in 6(a).
[Solution]: (3) By substitution

\[
\left( \left( 1 - \frac{\delta}{1 + \delta} \right) \left( 1 + \frac{\delta}{1 + \delta} \right)^{1+2\delta} \right)^{-n}
\]

\[
\leq \left( \exp \left( -\epsilon + (1 - \epsilon)\left( \frac{\delta^2}{1 + \delta} + \delta \right) \right) \right)^{-n}
\]

\[
\leq \exp \left( -n \left( (1 - \epsilon)\delta^2 - \epsilon \right) \right)
\]